Introduction to Representation Theory
MT 2020
Problem Sheet 0
(mainly revision: please do these before the start of term)

1. The symmetric group S_3 acts on polynomials in 3 variables $\mathbb{R}[x_1, x_2, x_3]$ by permuting the indices:

$$\sigma \cdot f(x_1, x_2, x_3) := f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) \quad \text{for all} \quad \sigma \in S_3, f(x_1, x_2, x_3) \in \mathbb{R}[x_1, x_2, x_3].$$

   Let $d \geq 1$ be an integer.

   (i) Show that S_3 preserves the set $V_d \subset \mathbb{R}[x_1, x_2, x_3]$ of homogeneous polynomials of degree d.

   (ii) For each $\sigma \in S_3$, let $\rho_d(\sigma) : V_d \to V_d$ be the action of σ. Prove that $\rho_d(\sigma)$ is $\mathbb{R}$-linear.

   (iii) Calculate the matrices of $\rho_1((123))$ and $\rho_1((12))$ with respect to the basis $\{v_1, v_2, v_3\}$ for V_1, where $v_1 := x_3, v_2 := x_1 - x_2$, and $v_3 := x_1 + x_2$.

   (iv) Calculate the matrix of $\rho_2((123))$ with respect to the basis $\{x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2\}$ for V_2.

2. Calculate the conjugacy classes in the following finite groups G:

   (a) $G = D_8$ the dihedral group of order 8,

   (b) $G = A_4$ the alternating group of order 12,

   (c) $G = S_5$ the symmetric group of order 120.

   Let $n \geq 1$ be an integer, and let $\sigma \in A_n$ be an even permutation. When does its S_n-conjugacy class equal its A_n-conjugacy class?

3. Let V_4 be the subgroup of S_4 generated by the double transpositions. By considering the conjugation action of S_4 on $V_4 \backslash \{1\}$ or otherwise, prove that S_4/V_4 is isomorphic to S_3.

4. Find all solutions to the system of equations $z + w = -2$ and $|z|^2 + |w|^2 = 2$ for $z, w \in \mathbb{C}$.

5. (a) Let G be a finite group. Show that there is a positive integer n such that G is isomorphic to a subgroup of $\mathrm{GL}_n(\mathbb{C})$.

   (b) Can you find a finite group H which *cannot* be isomorphic to a subgroup of $\mathrm{GL}_2(\mathbb{C})$?