Introduction to Representation Theory

MT 2020

Problem Sheet 1

Throughout this sheet, k denotes a field and G denotes a finite group.

1. Let $g \in \mathrm{GL}(V)$ be an element of finite order and suppose that k is algebraically closed. Prove that g is diagonalisable whenever $\mathrm{char}(k) = 0$. Does this result also hold for fields of positive characteristic?

2. The symmetric group S_n acts on $X := \{x_1, \cdots, x_n\}$ by permuting indices: $\sigma \cdot x_i = x_{\sigma(i)}$ for all $\sigma \in S_n$ and all i. Find all S_n-stable subspaces of the permutation representation $\rho : S_n \to \mathrm{GL}(kX)$.

3. Show that in Example 1.17, the G-stable subspace $\langle v_1 \rangle$ has no G-stable complement in $V = \langle v_1, v_2 \rangle$.

4. Let X be a G-set and suppose that the permutation representation $\rho : G \to \mathrm{GL}(kX)$ is irreducible. Prove that the G-action on X must be transitive. Is the converse true?

5. For each conjugacy class C in G, define its *conjugacy class sum* to be $\widehat{C} := \sum_{x \in C} x \in kG$. Prove that the conjugacy class sums form a basis for $Z(kG)$.

6. Suppose that $A = M_n(k)$ be the ring of $n \times n$ matrices with entries in k and let $V := k^n$ be the natural left A-module of $n \times 1$ column vectors.

   (a) Prove that V is a simple A-module.

   (b) Prove that A has no nonzero proper two-sided ideals.

   (c) Exhibit explicit simple left ideals $L_1, \cdots, L_n$ of A such that $A = L_1 \oplus \cdots \oplus L_n$.

   (d) Is the decomposition you found in (iii) unique? Justify your answer.

7. Let A be k-algebra for some field k and let M be a finite dimensional A-module. A *composition series* for M is a finite ascending chain

   $$\{0\} = M_0 < M_1 < M_2 < \cdots < M_n = M$$

   such that each subquotient M_k/M_{k-1} is a simple A-module for each $k = 1, \cdots, n$. These subquotients are called *composition factors*. Prove the *Jordan-Hölder Theorem*, which states that if

   $$\{0\} = N_0 < N_1 < N_2 < \cdots < N_m = M$$

   is another composition series for M, then necessarily $m = n$ and there exists a permutation $\sigma \in S_n$ together with A-module isomorphisms

   $$M_k/M_{k-1} \xrightarrow{\cong} N_{\sigma(k)}/N_{\sigma(k)-1} \quad \text{for all} \quad k = 1, \cdots, n.$$

   Deduce that G has only finitely many irreducible representations, up to isomorphism.