Introduction to Representation Theory

MT 2020

Problem Sheet 2

Throughout this sheet, k denotes a field, A denotes a ring and G denotes a finite group.

1. For each $a \in A$, let $r_a : A \to A$ be the left A-linear map given by $r_a(b) = ba$ for each $b \in A$. Prove that the map $r : A^{\mathrm{op}} \to \mathrm{End}_A(A)$ given by $r(a) = r_a$ is an isomorphism of rings.

2. (a) Suppose that $|G| \neq 0$ in k and let $e := \frac{1}{|G|} \sum_{g \in G} g \in kG$. Prove that e is a central idempotent.

   (b) Let $G = C_3 = \langle x \rangle$ be a cyclic group of order 3. Suppose that $\mathrm{char}(k) \neq 3$ and that k contains a primitive cube root of unity ω. Find an explicit isomorphism of k-algebras $k \times k \times k \xrightarrow{\cong} kC_3$.

3. (a) Prove that every representation $\rho : G \to \mathrm{GL}(V)$ extends to a k-algebra homomorphism $\widetilde{\rho} : kG \to \mathrm{End}_k(V)$.

   (b) Let $G = S_3$ and let $\rho : G \to \mathrm{GL}(W)$ be the degree 2 representation from Example 1.20. Prove that $\widetilde{\rho} : kG \to \mathrm{End}_k(W)$ is surjective, provided $\mathrm{char}(k) \neq 3$.

   (c) Assume that the characteristic of k is not 2 or 3. Using part (b), prove that there is an isomorphism of k-algebras

   $$kS_3 \xrightarrow{\cong} k \times k \times M_2(k).$$

   Does such an isomorphism exist when $\mathrm{char}(k) = 3$?

4. Find an example of a ring A that contains a field F such that A is *not* an F-algebra.

5. Let V be an A-module, let $D := \mathrm{End}_A(V)$ and let $n \geq 1$.

   (a) Use the inclusion maps $\sigma_j : V \hookrightarrow V^n$ and the projection maps $\pi_j : V^n \twoheadrightarrow V$ $(j = 1, \cdots, n)$ to construct an explicit ring isomorphism $M_n(D) \xrightarrow{\cong} \mathrm{End}_A(V^n)$.

   (b) Prove that $M_n(S)^{\mathrm{op}}$ is isomorphic to $M_n(S^{\mathrm{op}})$ for any ring S.

6. Let V be a finite dimensional kG-module.

   (a) Let W be a one-dimensional kG-module. Prove that $V \otimes W$ is simple if and only if V is simple.

   (b) Prove that V is simple if and only if V^* is simple.

7. Recall the maps $\alpha : V^* \otimes W \to \mathrm{Hom}(V, W)$ and $\beta : \mathrm{Hom}(V, W) \to V^* \otimes W$ from Lemma 4.11.

   (a) Prove that $\beta \circ \alpha = 1_{V^* \otimes W}$.

   (b) Prove that α is a homomorphism of kG-modules.

8. (*Optional.*) Let U, V, W be finite dimensional kG-modules. Prove $\mathrm{Hom}(U \otimes V, W)$ is isomorphic to $\mathrm{Hom}(U, \mathrm{Hom}(V, W))$ as kG-modules.