Introduction to Representation Theory

MT 2020

Problem Sheet 3

Throughout this sheet, G denotes a finite group.

1. Let V be a finite dimensional $\mathbb{C}G$-module and let $g \in G$. Prove that
$$\chi_{S^2V}(g) = \frac{1}{2}(\chi_V(g)^2 + \chi_V(g^2)) \quad \text{and} \quad \chi_{\Lambda^2V}(g) = \frac{1}{2}(\chi_V(g)^2 - \chi_V(g^2)).$$

2. Show that every group homomorphism from G to an abelian group A is trivial on the commutator subgroup G' and hence factors through G/G'. Show that if N is a normal subgroup of G with G/N abelian, then $G' \leqslant N$.

3. Let k be an algebraicaly closed field.
   (a) Suppose that G is abelian. Prove that every simple kG-module is one-dimensional.
   (b) Prove that the converse holds provided that $|G| \neq 0$ in k.
   (c) Deduce from (a) that G has precisely $|G : G'|$ complex linear characters.

4. Calculate the character of the representation $\rho : S_4 \to \mathrm{GL}_3(\mathbb{R})$ from Example 1.3(d). Let $V = \mathbb{C}^3$ be the $\mathbb{C}S_4$-module obtained by viewing ρ as a complex representation $S_4 \to \mathrm{GL}_3(\mathbb{C})$. Decompose $V \otimes V$ as a direct sum of irreducible representations.

5. (a) Let χ be a character of G. Show that $\{g \in G : \chi(g) = \chi(1)\}$ is a normal subgroup of G.
   (b) Prove that G is simple if and only if $\chi(g) \neq \chi(1)$ for every $g \neq 1$ and every irreducible $\chi \neq \mathbb{1}$.

6. Let G act on a finite set X and consider the permutation module $V := \mathbb{C}X$.
   (a) Let $g \in G$. Prove that $\chi_V(g) = |\operatorname{Fix}_X(g)|$ where $\operatorname{Fix}_X(g) := \{x \in X : g \cdot x = x\}$.
   (b) Prove that $\sum_{g \in G} \chi_V(g) = r|G|$, where r is the number of G-orbits on X.
   (c) Suppose now that the action of G on X is 2-*transitive*, that is G has two orbits acting on $X \times X$ in the action defined by $g \cdot (x_1, x_2) = (g \cdot x_1, g \cdot x_2)$. Show that $\sum_{g \in G} \chi_V(g)^2 = 2|G|$ and deduce that $V = \mathbb{1} \oplus W$ for some simple submodule W of V.

7. Find the character tables of the quaternion group Q_8 and the dihedral group D_8 of order 8. Does the character table determine the group up to isomorphism?

8. Let H be another finite group whose character table is equal to the character table of G. Prove that $|G'| = |H'|$ and that $|Z(G)| = |Z(H)|$.

9. (a) Let χ be a character of G. Show that $\chi(g^{-1}) = \overline{\chi(g)}$ for all $g \in G$.
   (b) Show that $g \in G$ is conjugate to g^{-1} if and only if $\chi(g) \in \mathbb{R}$ for every character χ of G.