Introduction to Representation Theory

MT 2020

Problem Sheet 4

1. Find the character table of the alternating group A_5. It may be helpful to remember that A_5 acts as a group of rotations of the regular icosahedron.

2. Let G be a finite group with an irreducible representation $\rho : G \to \mathrm{GL}_2(\mathbb{C})$.

   (a) Prove that G has an element a of order 2.

   (b) For a as above show that either $\det \rho(a) \neq 1$ or else $\rho(a)$ is central in $\mathrm{GL}_2(\mathbb{C})$.

   (c) Deduce that a finite simple group cannot have an irreducible representation of degree 2.

3. Let G be a finite group and suppose that V is a simple $\mathbb{C}G$-module. Define

   (a) Prove that $e_V = \frac{\dim V}{|G|} \sum_{g \in G} \overline{\chi_V(g)}\, g$ is an element of the centre of $\mathbb{C}G$.

   (b) Let V' be another simple $\mathbb{C}G$-module. Prove that e_V kills V' if V' is not isomorphic to V, and that e_V acts as the identity on V.

   (c) Let $V_1, \cdots, V_r$ be the simple $\mathbb{C}G$-modules (up to isomorphism) and let $e_i := e_{V_i}$ for $i = 1, \cdots, r$. Prove that $e_i e_j = \delta_{i,j} e_i$ for all $i, j = 1, \cdots, r$, and that $e_1 + \cdots + e_r = 1$.

4. A conjugacy class g^G of a finite group G is called *real* if g is conjugate to g^{-1}. A character χ of G is called *real* if $\chi(g) \in \mathbb{R}$ for all $g \in G$. By considering the vector space

$$V := \{f : G \to \mathbb{C} : f(g) = f(h^{-1}gh) = f(g^{-1}) \quad \text{for all} \quad g, h \in G\}$$

   or otherwise, prove that the number of real conjugacy classes in G is equal to the number of irreducible real characters.

5. Prove that every finite group has a faithful representation. Which finite abelian groups have a faithful irreducible representation?

6. Let H be a cyclic subgroup of $G := S_4$ and let $\varphi : H \to \mathbb{C}^\times$ be a faithful linear character. Write $\mathrm{Ind}_H^G \varphi$ as a sum of irreducible characters of G when (a) $H = \langle(1234)\rangle$, and (b) $H = \langle(123)\rangle$.

7. (a) Let V be a simple $\mathbb{C}G$-module and let W be a simple $\mathbb{C}H$-module. Construct a linear $G \times H$-action on $V \otimes W$ and prove that the resulting $\mathbb{C}(G \times H)$-module is simple.

   (b) Let V be a simple $\mathbb{C}G$-module and let Z be the centre of G. Show that for each $m \geq 1$, the subgroup $D_m := \{(z_1, \cdots, z_m) \in Z^m : z_1 \cdot \cdots \cdot z_m = 1\}$ of Z^m acts trivially on $V^{\otimes m}$.

   (c) By considering large values of m, deduce that $\dim V$ divides $|G/Z|$.

8. (*Optional.*) Prove that induction is transitive: if k is a field and $J \subseteq H$ are subgroups of G, then

$$\operatorname{Ind}_H^G(\operatorname{Ind}_J^H V) \cong \operatorname{Ind}_J^G V$$

as kG-modules, for every kJ-module V.

9. (*Optional.*) Suppose that V is a faithful representation of G. Prove that every simple $\mathbb{C}G$-module W appears as a direct summand of some tensor power $V^{\otimes n}$ of V, by considering the infinite series

$$\sum_{n \geq 0} \langle \chi_W, \chi_{V^{\otimes n}} \rangle t^n$$

where t is an indeterminate.

10. (*Optional.*) Construct the character table of A_6 as follows.

   (a) Use the conjugation action of A_5 on its set of Sylow 5-subgroups to construct an injective homomorphism $\sigma : A_5 \to A_6$, and prove that its image contains no 3-cycles.

   (b) Use the left-multiplication action of A_6 on $A_6/\sigma(A_5)$ to construct an automorphism $\tau : A_6 \to A_6$, and prove that τ swaps the two conjugacy classes in A_6 consisting of elements of order 3.

   (c) Use the natural 2-transitive action on A_6 on $\{1, 2, 3, 4, 5, 6\}$ together with part (b) to write down two irreducible characters χ_2 and χ_3 of A_6, each of degree 5.

   (d) Use $\Lambda^2\chi_2$ and $\chi_2\chi_3$ and the Orthogonality Theorems to complete the character table of A_6.