B1.1 Logic

Jonathan Pila

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Slides by **J. Koenigsmann** with revisions
Main further reference:
D. Goldrei, “Propositional and Predicate Calculus: A Model of Argument”, Springer.

Mathematical Logic:

1. What is it about? It:

- provides a uniform, unambiguous **language** for mathematics

- makes precise what a **proof** is

- explains and guarantees **exactness, rigour**, and **certainty** in mathematics

- establishes **foundations** for mathematics

B1 (Foundations)
= B1.1 (Logic) + B1.2 (Set theory)

N.B.: The course does not teach you to think logically, but it explores what it *means* to think logically.

2. Historical motivation

- *19th cent.*:
  need for conceptual foundation in analysis:
  what is the correct notion of
  infinity, infinitesimal, limit, ...

- attempts to formalize mathematics:
  - *Frege*'s Begriffsschrift
  - *Cantor*'s **naive** set theory: a set is
  "any collection into a whole of definite and separate objects of our intuition or our thought...

- led to **Russell's paradox**:
  consider the set $R := \{S \text{ set } \mid S \notin S\}$

  $$R \in R \;\Rightarrow\; R \notin R \quad \text{contradiction}$$
  $$R \notin R \;\Rightarrow\; R \in R \quad \text{contradiction}$$

  $\rightsquigarrow$ *fundamental crisis in the foundations of mathematics*

3. Hilbert's Program (early C20)

1. find a uniform (formal) **language** for all mathematics

2. find a complete system of **inference rules/ deduction rules**

3. find a complete system of mathematical **axioms**

4. prove that the system 1.+2.+3. is **consistent**, i.e. does not lead to contradictions

⋆ **complete:** every mathematical sentence can be proved or disproved using 2. and 3.

⋆ 1., 2. and 3. should be **finitary/effective/computable/algorithmic** so, e.g., in 3. you can't take as axioms *the system of all true sentences in mathematics*

⋆ **idea:** any piece of information is of finte length

4. Progress on Hilbert's program

Step 1. is possible in the framework of
ZF = *Zermelo-Fraenkel set theory* or
ZFC = **ZF** + *Axiom of* **C***hoice*
(this is an empirical fact)
$\rightsquigarrow$ B1.2 Set Theory HT, Ax Set Th C1.4

Step 2. is possible in the framework of
1st-order logic:
Gödel's Completeness Theorem, 1929
$\rightsquigarrow$ B1.1 Logic - **this course**

Step 3. is not possible ($\rightsquigarrow$ C1.2):
Gödel's 1st Incompleteness Theorem:
there is no effective axiomatization
of arithmetic

Step 4. is not possible ($\rightsquigarrow$ C1.2):
Gödel's 2nd Incompleteness Theorem, (but..)

5. Decidability

Step 3. of Hilbert's program fails:
there is no effective axiomatization
for the entire body of mathematics

But: many important parts of mathematics are completely and effectively axiomatizable, they are **decidable**, i.e. there is an *algorithm = program = effective procedure* deciding whether a statement is true or false
$\rightsquigarrow$ allows proofs by computer

Example: Th($\mathbb{C}$) = the **1st-order theory** of $\mathbb{C}$
= all *algebraic* properties of $\mathbb{C}$:

Axioms = *field axioms*
+ *all non-constant polynomials have a zero*
+ *the characteristic is* 0 *(See §10.9.)*

Every algebraic property of $\mathbb{C}$ follows from these axioms (see §15).
Similarly for Th($\mathbb{R}$).
$\rightsquigarrow$ C1.1 Model Theory

6. Why *mathematical* logic?

1. Language and deduction rules are tailored for *mathematical objects* and *reasoning*

   N.B.: Logic tells you what a proof *is*, not how to *find* one

2. The *method* is mathematical:
   we develop logic as a *calculus*: a kind of "algebra with formulas". That is:
   Mathematical Logic is mathematics!
   not meta-mathematics or philosophy;
   we won't consider ontological questions like *what is a number?*

3. Mathematical Logic has *applications* to other areas of mathematics, such as Algebra, Topology, and Number Theory, but also to Theoretical Computer Science

7. This course...

1. ... is devoted to developing the framework of **first-order predicate calculus**...

2. (Sounds a bit niche but this is the most widely used system. It is very well adapted to many parts of mathematics-in-practice, and the basis of **model theory**.)

3. ...and to proving the key **Completeness Theorem** of Gödel 1929: the associated deductive system proves "everything it could".

4. We do it first for a very simple system, the **propositional calculus** (Part I), and then move to **predicate calculus** (Part II).