### PART I: Propositional Calculus

# 1. The language of propositional calculus

... is a very coarse language with limited expressive power

... allows you to break a complicated sentence down into its subclauses, but not any further

... will be refined in PART II *Predicate Calculus*, the true language of first order logic

... is nevertheless well suited for entering formal logic (and Part II will build on it)

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#### 1.1 Propositional variables

- all mathematical disciplines use variables,
  e.g. x, y for real numbers
  or z, w for complex numbers
  or α, β for angles etc.
- in logic we introduce variables p<sub>0</sub>, p<sub>1</sub>, p<sub>2</sub>, ...
  for sentences (*propositions*)
- we don't care what these propositions say, only their *logical properties* count, i.e. whether they are *true* or *false* (when we use *variables* for real numbers, we also don't care about *particular* numbers)

## 1.2 The alphabet of propositional calculus

consists of the following symbols:

the propositional variables  $p_0, p_1, \ldots, p_n, \ldots$ 

**negation**  $\neg$  - the unary connective *not* 

four binary connectives  $\rightarrow$ ,  $\land$ ,  $\lor$ ,  $\leftrightarrow$ *implies, and, or* and *if and only if* respectively

**two punctuation marks** ( and ) *left parenthesis* and *right parenthesis* 

This alphabet is denoted by  $\mathcal{L}$ . Note that these are *abstract symbols*. Note also that we use  $\rightarrow$ , and not  $\Rightarrow$ .

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#### 1.3 Strings

#### • A string (from *L*)

is any finite sequence of symbols from  ${\cal L}$  placed one after the other - no gaps

#### • Examples

(i) 
$$\to p_{17}()$$
  
(ii)  $((p_0 \land p_1) \to \neg p_2)$   
(iii)  $)) \neg )p_{32}$ 

• The **length** of a string is the number of symbols in it.

So the strings in the examples have length 4, 10, 5 respectively.

(A propositional variable has length 1.)

• we now single out from all strings those which make grammatical sense (*formulas*)

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#### 1.4 Formulas

The notion of a **formula of**  $\mathcal{L}$  is defined (*re-cursively*) by the following rules:

I. every propositional variable is a formula

**II.** if the string A is a formula then so is  $\neg A$ 

**III.** if the strings A and B are both formulas then so are the strings

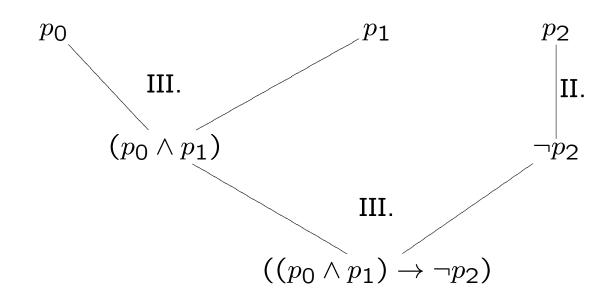
$(A \rightarrow B)$	read $A \ implies \ B$
$(A \wedge B)$	read A and B
$(A \lor B)$	read A or B
$(A \leftrightarrow B)$	read A if and only if B

**IV.** Nothing else is a formula,

i.e. a string  $\phi$  is a formula if and only if  $\phi$  can be obtained from propositional variables by finitely many applications of the *formation rules* II. and III.

#### Examples

• the string  $((p_0 \land p_1) \rightarrow \neg p_2)$  is a formula (Example (ii) in 1.3) *Proof:* 



- Parentheses are important, e.g.  $(p_0 \land (p_1 \rightarrow \neg p_2))$  is a different formula and  $p_0 \land (p_1 \rightarrow \neg p_2)$  is not a formula at all
- the strings  $\rightarrow p_{17}()$  and  $)) \neg p_{32}$  from Example (i) and (iii) in 1.3 are not formulas: this follows from the following Lemma:

 $\Box$ 

**Lemma** If  $\phi$  is a formula then

- either  $\phi$  is a propositional variable

- or the first symbol of  $\phi$  is  $\neg$ 

- or the first symbol of  $\phi$  is (.

*Proof:* Induction on n := the length of  $\phi$ :

n = 1: then  $\phi$  is a propositional variable any formula obtained via formation rules (II. and III.) has length > 1.

Suppose the lemma holds for all formulas of length  $\leq n$ .

Let  $\phi$  have length n+1

⇒  $\phi$  is not a propositional variable  $(n + 1 \ge 2)$ ⇒ either  $\phi$  is  $\neg \psi$  for some formula  $\psi$  - so  $\phi$ begins with  $\neg$ 

or  $\phi$  is  $(\psi_1 \star \psi_2)$  for some  $\star \in \{\rightarrow, \land, \lor, \leftrightarrow\}$  and some formulas  $\psi_1$ ,  $\psi_2$  - so  $\phi$  begins with (.  $\Box$ 

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#### The unique readability theorem

A formula can be constructed in only one way: For each formula  $\phi$  **exactly one** of the following holds

(a)  $\phi$  is  $p_i$  for some unique  $i \in \mathbf{N}$ ;

(b)  $\phi$  is  $\neg \psi$  for some **unique** formula  $\psi$ ;

(c)  $\phi$  is  $(\psi \star \chi)$  for some **unique** pair of formulas  $\psi$ ,  $\chi$  and a **unique** binary connective  $\star \in \{\rightarrow, \land, \lor, \leftrightarrow\}$ .

*Proof:* Problem sheet #1.

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