PART I: Propositional Calculus

1. The language of propositional calculus

... is a very coarse language with limited expressive power

... allows you to break a complicated sentence down into its subclauses, but not any further

... will be refined in PART II *Predicate Calculus*, the true language of first order logic

... is nevertheless well suited for entering formal logic (and Part II will build on it)

1.1 Propositional variables

- all mathematical disciplines use variables, e.g. x, y for real numbers or z, w for complex numbers or α, β for angles etc.

- in logic we introduce variables $p_0, p_1, p_2, \ldots$ for sentences (*propositions*)

- we don't care what these propositions say, only their *logical properties* count, i.e. whether they are *true* or *false* (when we use *variables* for real numbers, we also don't care about *particular* numbers)

1.2 The alphabet of propositional calculus

consists of the following symbols:

the propositional variables $p_0, p_1, \ldots, p_n, \ldots$

negation $\neg$ - the unary connective *not*

four binary connectives $\rightarrow$, $\wedge$, $\vee$, $\leftrightarrow$
implies, and, or and *if and only if* respectively

two punctuation marks (and)
left parenthesis and *right parenthesis*

This alphabet is denoted by $\mathcal{L}$.
Note that these are *abstract symbols*.
Note also that we use $\rightarrow$, and not $\Rightarrow$.

1.3 Strings

- A **string (from $\mathcal{L}$)** is any finite sequence of symbols from $\mathcal{L}$ placed one after the other - no gaps

- **Examples**

$$\begin{array}{rl} \text{(i)} & \to p_{17}() \\ \text{(ii)} & ((p_0 \wedge p_1) \to \neg p_2) \\ \text{(iii)} &))\neg)p_{32} \end{array}$$

- The **length** of a string is the number of symbols in it.
  So the strings in the examples have length $4, 10, 5$ respectively.
  (A propositional variable has length 1.)

- we now single out from all strings those which make grammatical sense (*formulas*)

1.4 Formulas

The notion of a **formula of $\mathcal{L}$** is defined (*recursively*) by the following rules:

I. every propositional variable is a formula

II. if the string A is a formula then so is $\neg A$

III. if the strings A and B are both formulas then so are the strings

$$\begin{array}{rl} (A \to B) & \text{read } A \textit{ implies } B \\ (A \wedge B) & \text{read } A \textit{ and } B \\ (A \vee B) & \text{read } A \textit{ or } B \\ (A \leftrightarrow B) & \text{read } A \textit{ if and only if } B \end{array}$$

IV. Nothing else is a formula,
i.e. a string ϕ is a formula if and only if ϕ can be obtained from propositional variables by finitely many applications of the *formation rules* II. and III.

Examples

- the string $((p_0 \wedge p_1) \to \neg p_2)$ is a formula (Example (ii) in 1.3)

  Proof:

$$
\begin{tikzcd}
p_0 \arrow[dr, dash] & & p_1 \arrow[dl, dash, "\mathrm{III.}"'] & & p_2 \arrow[d, dash, "\mathrm{II.}"] \\
& (p_0 \wedge p_1) \arrow[drr, dash] & & & \neg p_2 \arrow[dl, dash, "\mathrm{III.}"'] \\
& & & ((p_0 \wedge p_1) \to \neg p_2) &
\end{tikzcd}
$$

$\square$

- Parentheses are important, e.g. $(p_0 \wedge (p_1 \to \neg p_2))$ is a different formula and $p_0 \wedge (p_1 \to \neg p_2)$ is not a formula at all

- the strings $\to p_{17}()$ and $))\neg)p_{32}$ from Example (i) and (iii) in 1.3 are not formulas: this follows from the following Lemma:

Lemma *If ϕ is a formula then*
- either ϕ is a propositional variable
- or the first symbol of ϕ is $\neg$
- or the first symbol of ϕ is (.

Proof: Induction on $n :=$ the length of ϕ:

$\mathbf{n = 1}$: then ϕ is a propositional variable - any formula obtained via formation rules (II. and III.) has length > 1.

Suppose the lemma holds for all formulas of length $\leq n$.
Let ϕ have length $n + 1$
$\Rightarrow$ ϕ is not a propositional variable ($n + 1 \geq 2$)
$\Rightarrow$ either ϕ is $\neg\psi$ for some formula ψ - so ϕ begins with $\neg$

or ϕ is $(\psi_1 \star \psi_2)$ for some $\star \in \{\rightarrow, \wedge, \vee, \leftrightarrow\}$ and some formulas ψ_1, ψ_2 - so ϕ begins with (. $\square$

The unique readability theorem

A formula can be constructed in only one way: For each formula ϕ **exactly one** *of the following holds*

(a) ϕ is p_i for some unique $i \in \mathbf{N}$;

(b) ϕ is $\neg\psi$ for some **unique** formula ψ;

(c) ϕ is $(\psi \star \chi)$ for some **unique** pair of formulas ψ, χ and a **unique** binary connective $\star \in \{\to, \wedge, \vee, \leftrightarrow\}$.

Proof: Problem sheet #1.