6. A deductive system for propositional calculus

- We defined **logical consequence**, $\Gamma \models \phi$: whenever (every formula of) Γ is true so is ϕ. Then ϕ "follows" from $\Gamma \ldots$

- ... but we don't know how to give a **proof** of ϕ from the **hypotheses** Γ.

- A **proof** should be a finite sequence $\phi_1, \phi_2, \ldots, \phi_n$ of statements such that
  - either $\phi_i \in \Gamma$
  - or ϕ_i is some **axiom** (which should *clearly* be true)
  - or ϕ_i should follow from previous ϕ_j's by some **rule of inference**
  - AND $\phi = \phi_n$

6.1 Definition
Let $\mathcal{L}_0 := \mathcal{L}[\{\neg, \to\}]$ (which is an adequate language). Then the **system** L_0 consists of the following axioms and rules:

Axioms
An **axiom** of L_0 is any formula of the following form ($\alpha, \beta, \gamma \in \text{Form}(\mathcal{L}_0)$):

A1 $(\alpha \to (\beta \to \alpha))$

A2 $(((\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma)))$

A3 $((\neg\beta \to \neg\alpha) \to (\alpha \to \beta))$

Rules of inference
Only one: **modus ponens**
(for any $\alpha, \beta \in \text{Form}(\mathcal{L}_0)$)
MP From α and $(\alpha \to \beta)$ infer β.

6.2 Definition
For any $\Gamma \subseteq \text{Form}(\mathcal{L}_0)$ we say that α is **deducible** (or **provable**) from the hypotheses Γ if there is a finite sequence $\alpha_1, \ldots, \alpha_m \in \text{Form}(\mathcal{L}_0)$ such that for each $i = 1, \ldots, m$ either

(a) α_i is an axiom, or
(b) $\alpha_i \in \Gamma$, or
(c) there are $j < k < i$ such that α_i follows from α_j, α_k by MP,
i.e. $\alpha_j = (\alpha_k \to \alpha_i)$ or $\alpha_k = (\alpha_j \to \alpha_i)$

AND

(d) $\alpha_m = \alpha$.

The sequence $\alpha_1, \ldots, \alpha_m$ is then called a **proof** or **deduction** or **derivation** of α from Γ.

Write $\Gamma \vdash \alpha$.

If $\Gamma = \emptyset$ write $\vdash \alpha$ and say that α is a **theorem** (of the system L_0).

6.3 Example For any $\phi \in \mathrm{Form}(\mathcal{L}_0)$

$$(\phi \to \phi)$$

is a theorem of L_0.

Proof:

α_1 $(\phi \to (\phi \to \phi))$
[A1 with $\alpha = \beta = \phi$]
α_2 $(\phi \to ((\phi \to \phi) \to \phi))$
[A1 with $\alpha = \phi$, $\beta = (\phi \to \phi)$]
α_3 $((\phi \to ((\phi \to \phi) \to \phi)) \to$
$\to ((\phi \to (\phi \to \phi)) \to (\phi \to \phi)))$
[A2 with $\alpha = \phi$, $\beta = (\phi \to \phi)$, $\gamma = \phi$]
α_4 $((\phi \to (\phi \to \phi)) \to (\phi \to \phi))$
[MP α_2, α_3]
α_5 $(\phi \to \phi)$
[MP α_1, α_4]

Thus, $\alpha_1, \alpha_2, \ldots, \alpha_5$ is a deduction of $(\phi \to \phi)$ in L_0.

$\square$

6.4 Example

For any $\phi, \psi \in \text{Form}(\mathcal{L}_0)$:

$$\{\phi, \neg\phi\} \vdash \psi$$

Proof:

α_1 $(\neg\phi \to (\neg\psi \to \neg\phi))$
[A1 with $\alpha = \neg\phi, \beta = \neg\psi$]

α_2 $\neg\phi$ [$\in \Gamma$]

α_3 $(\neg\psi \to \neg\phi)$ [MP α_1, α_2]

α_4 $((\neg\psi \to \neg\phi) \to (\phi \to \psi))$
[A3 with $\alpha = \phi, \beta = \psi$]

α_5 $(\phi \to \psi)$ [MP α_3, α_4]

α_6 ϕ [$\in \Gamma$]

α_7 ψ [MP α_5, α_6]

$\square$

6.5 The Soundness Theorem for L_0

L_0 is **sound**, *i.e. for any $\Gamma \subseteq Form(\mathcal{L}_0)$ and for any $\alpha \in Form(\mathcal{L}_0)$:*

$$\textit{if } \Gamma \vdash \alpha \textit{ then } \Gamma \models \alpha.$$

In particular, any theorem of L_0 is a tautology.

Proof:
Assume $\Gamma \vdash \alpha$ and let $\alpha_1, \alpha_2, \ldots, \alpha_m = \alpha$ be a deduction of α in L_0.
Let v be any valuation such that $\tilde{v}(\phi) = T$ for all $\phi \in \Gamma$.

We have to show that $\tilde{v}(\alpha) = T$.

We show by induction on $i \leq m$ that

$$\tilde{v}(\alpha_1) = \ldots = \tilde{v}(\alpha_i) = T \quad (\star)$$

i = 1
either α_1 is an axiom, so $\widetilde{v}(\alpha_1) = T$ or $\alpha_1 \in \Gamma$, so, by hypothesis, $\widetilde{v}(\alpha_1) = T$.

Induction step
Suppose ($\star$) is true for some $i < m$.
Consider α_{i+1}.

Either α_{i+1} is an axiom or $\alpha_{i+1} \in \Gamma$, so $\widetilde{v}(\alpha_{i+1}) = T$ as above,

or else there are $j \neq k < i + 1$ such that $\alpha_j = (\alpha_k \to \alpha_{i+1})$.

By induction hypothesis

$$\widetilde{v}(\alpha_k) = \widetilde{v}(\alpha_j) = \widetilde{v}((\alpha_k \to \alpha_{i+1})) = T.$$

But then, by tt $\to$, $\widetilde{v}(\alpha_{i+1}) = T$ (since $T \to F$ is F).

$\square$

For the proof of the converse

Completeness Theorem

$$\text{\textit{If} } \Gamma \models \alpha \text{ \textit{then} } \Gamma \vdash \alpha.$$

we first prove

6.6 The Deduction Theorem for L_0

For any $\Gamma \subseteq$ *Form*$(\mathcal{L}_0)$ *and*
for any $\alpha, \beta \in$ *Form*$(\mathcal{L}_0)$*:*

$$\text{\textit{if} } \Gamma \cup \{\alpha\} \vdash \beta \text{ \textit{then} } \Gamma \vdash (\alpha \to \beta).$$