

7. Consistency, Completeness and Compactness

7.1 Definition

Let $\Gamma \subseteq \text{Form}(\mathcal{L}_0)$. Γ is said to be **consistent** (or $\mathcal{L}_0$ -consistent) if for *no* formula α both $\Gamma \vdash \alpha$ and $\Gamma \vdash \neg\alpha$.

Otherwise Γ is **inconsistent**.

E.g. $\emptyset$ is consistent: by soundness theorem, α and $\neg\alpha$ are never simultaneously true.

7.2. Lemma

$\Gamma \cup \{\neg\phi\}$ is inconsistent iff $\Gamma \vdash \phi$.

(In part., if $\Gamma \not\vdash \phi$ then $\Gamma \cup \{\neg\phi\}$ is consistent).

Proof: ‘ $\Leftarrow$ ’:

$$\Gamma \vdash \phi \Rightarrow \left. \begin{array}{l} \Gamma \cup \{\neg\phi\} \vdash \phi \\ \Gamma \cup \{\neg\phi\} \vdash \neg\phi \end{array} \right\} \Rightarrow \Gamma \cup \{\neg\phi\} \text{ is inconsistent}$$

‘ $\Rightarrow$ ’:

$$\left. \begin{array}{l} \Gamma \cup \{\neg\phi\} \vdash \alpha \\ \Gamma \cup \{\neg\phi\} \vdash \neg\alpha \end{array} \right\} \Rightarrow_{6.11} \left. \begin{array}{l} \Gamma \cup \{\neg\phi\} \vdash_{SQ} \alpha \\ \Gamma \cup \{\neg\phi\} \vdash_{SQ} \neg\alpha \end{array} \right\}$$

$$\Rightarrow_{PC} \Gamma \vdash_{SQ} \phi \Rightarrow_{6.11} \Gamma \vdash \phi$$

□

7.3 Lemma

Suppose Γ is consistent and $\Gamma \vdash \phi$.
Then $\Gamma \cup \{\phi\}$ is consistent.

Proof: Suppose not, i.e. for some α

$$\left. \begin{array}{l} \Gamma \cup \{\phi\} \vdash \alpha \\ \Gamma \cup \{\phi\} \vdash \neg\alpha \end{array} \right\} \Rightarrow_{\text{DT}} \left. \begin{array}{l} \Gamma \vdash (\phi \rightarrow \alpha) \\ \Gamma \vdash (\phi \rightarrow \neg\alpha) \end{array} \right\} \xRightarrow[\text{MP}]{\Gamma \vdash \phi} \Rightarrow \begin{array}{l} \Gamma \vdash \alpha \\ \Gamma \vdash \neg\alpha \end{array} \quad \text{\textcancel{X}}$$

□

7.4 Definition

$\Gamma \subseteq \text{Form}(\mathcal{L}_0)$ is **maximal consistent** if

- (i) Γ is consistent, and
- (ii) for every ϕ , either $\Gamma \vdash \phi$ or $\Gamma \vdash \neg\phi$.

Note: This is equivalent to saying that for every ϕ , if $\Gamma \cup \{\phi\}$ is consistent then $\Gamma \vdash \phi$.

Proof: Exercise

7.5 Lemma

Suppose Γ is maximal consistent.

Then for every $\psi, \chi \in \text{Form}(\mathcal{L}_0)$

(a) $\Gamma \vdash \neg\psi$ iff $\Gamma \not\vdash \psi$

(b) $\Gamma \vdash (\psi \rightarrow \chi)$ iff either $\Gamma \vdash \neg\psi$ or $\Gamma \vdash \chi$.

Proof:

(a) ‘ $\Rightarrow$ ’: by consistency

‘ $\Leftarrow$ ’: by maximality

(b) ‘ $\Rightarrow$ ’: Suppose $\Gamma \not\vdash \neg\psi$ and $\Gamma \not\vdash \chi$

$\Rightarrow \Gamma \vdash \psi$ and $\Gamma \vdash \neg\chi$

$\Gamma \vdash (\psi \rightarrow \chi) \Rightarrow_{\text{MP}} \Gamma \vdash \chi \quad \text{\textcancel{X}}$

‘ $\Leftarrow$ ’: Suppose $\Gamma \vdash \neg\psi$

$\Gamma \vdash (\neg\psi \rightarrow (\psi \rightarrow \chi))$ - Problems # 2, (5)(i)

$\Rightarrow_{\text{MP}} \Gamma \vdash (\psi \rightarrow \chi)$

Suppose $\Gamma \vdash \chi$

$\Gamma \vdash (\chi \rightarrow (\psi \rightarrow \chi))$ - Axiom A1

$\Rightarrow_{\text{MP}} \Gamma \vdash (\psi \rightarrow \chi)$

□

7.6 Theorem

*Suppose Γ is maximal consistent.
Then Γ is satisfiable.*

Proof:

For each i , $\Gamma \vdash p_i$ or $\Gamma \vdash \neg p_i$ (by maximality),
but not both (by consistency)

Define a valuation v by

$$v(p_i) = \begin{cases} T & \text{if } \Gamma \vdash p_i \\ F & \text{if } \Gamma \vdash \neg p_i \end{cases}$$

Claim: for all $\phi \in \text{Form}(\mathcal{L}_0)$:

$$\tilde{v}(\phi) = T \text{ iff } \Gamma \vdash \phi$$

Proof by induction on the length n of ϕ :

n=1:

Then $\phi = p_i$ for some i , and so, by def. of v ,

$$\tilde{v}(p_i) = T \text{ iff } \Gamma \vdash p_i.$$

IH: Claim true for all $i \leq n$.

Now assume $\text{length}(\phi) = n+1$

Case 1: $\phi = \neg\psi$ ($\Rightarrow \text{length}(\psi) = n$)

$$\begin{aligned} \tilde{v}(\phi) = T & \text{ iff } \tilde{v}(\psi) = F & \text{tt } \neg \\ & \text{iff } \Gamma \not\vdash \psi & \text{IH} \\ & \text{iff } \Gamma \vdash \neg\psi & 7.5(a) \\ & \text{iff } \Gamma \vdash \phi \end{aligned}$$

Case 2: $\phi = (\psi \rightarrow \chi)$

($\Rightarrow \text{length}(\psi), \text{length}(\chi) \leq n$)

$$\begin{aligned} \tilde{v}(\phi) = T & \text{ iff } \tilde{v}(\psi) = F \text{ or } \tilde{v}(\chi) = T & \text{tt } \rightarrow \\ & \text{iff } \Gamma \not\vdash \psi \text{ or } \Gamma \vdash \chi & \text{IH} \\ & \text{iff } \Gamma \vdash \neg\psi \text{ or } \Gamma \vdash \chi & 7.5(a) \\ & \text{iff } \Gamma \vdash (\psi \rightarrow \chi) & 7.5(b) \\ & \text{iff } \Gamma \vdash \phi \end{aligned}$$

So $\tilde{v}(\phi) = T$ for all $\phi \in \Gamma$, i.e. v satisfies Γ .

□

7.7 Theorem

Suppose Γ is consistent. Then there is a maximal consistent Γ' such that $\Gamma \subseteq \Gamma'$.

Proof:

$\text{Form}(\mathcal{L}_0)$ is countable, say

$$\text{Form}(\mathcal{L}_0) = \{\phi_1, \phi_2, \phi_3, \dots\}.$$

Construct consistent sets

$$\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \dots$$

as follows: $\Gamma_0 := \Gamma$.

Having constructed Γ_n consistently, let

$$\Gamma_{n+1} := \begin{cases} \Gamma_n \cup \{\phi_{n+1}\} & \text{if } \Gamma_n \vdash \phi_{n+1} \\ \Gamma_n \cup \{\neg\phi_{n+1}\} & \text{if } \Gamma_n \not\vdash \phi_{n+1} \end{cases}$$

Then Γ_{n+1} is consistent by 7.3 and 7.2.

Now let $\Gamma' := \bigcup_{n=0}^{\infty} \Gamma_n$.

Then Γ' is consistent:

Any proof of $\Gamma' \vdash \alpha$ and $\Gamma' \vdash \neg\alpha$ would use only finitely many formulas from Γ' , so for some n , $\Gamma_n \vdash \alpha$ and $\Gamma_n \vdash \neg\alpha$ – contradicting the consistency of Γ_n .

Finally, Γ' is maximal (even in a stronger sense): for all n , either $\phi_n \in \Gamma'$ or $\neg\phi_n \in \Gamma'$. $\square$

Note that the proof does not make use of Zorn's Lemma.

7.8 Corollary

If Γ is consistent then Γ is satisfiable.

Proof: 7.6 + 7.7 $\square$

7.9 The Completeness Theorem

If $\Gamma \models \phi$ then $\Gamma \vdash \phi$.

Proof:

Suppose $\Gamma \models \phi$, but $\Gamma \not\vdash \phi$.

$\Rightarrow$ by 7.2, $\Gamma \cup \{\neg\phi\}$ is consistent

$\Rightarrow$ by 7.8, there is some valuation v such that

$\tilde{v}(\psi) = T$ for all $\psi \in \Gamma \cup \{\neg\phi\}$

$\Rightarrow \tilde{v}(\psi) = T$ for all $\psi \in \Gamma$, but $\tilde{v}(\phi) = F$

$\Rightarrow \Gamma \not\models \phi$: contradiction. $\square$

7.10 Corollary

(7.9 Completeness + 6.5 Soundness)

$$\Gamma \models \phi \text{ iff } \Gamma \vdash \phi$$

7.11 The Compactness Theorem for L_0

$\Gamma \subseteq \text{Form}(\mathcal{L}_0)$ is satisfiable iff every finite subset of Γ is satisfiable.

Proof: ' $\Rightarrow$ ': obvious –

if $\tilde{v}(\psi) = T$ for all $\psi \in \Gamma$ then $\tilde{v}(\psi) = T$ for all $\psi \in \Gamma' \subseteq \Gamma$.

' $\Leftarrow$ ':

Suppose every finite $\Gamma' \subseteq \Gamma$ is satisfiable, but Γ is not.

Then, by 7.8, Γ is inconsistent, i.e. $\Gamma \vdash \alpha$ and $\Gamma \vdash \neg\alpha$ for some α .

But then, for some *finite* $\Gamma' \subseteq \Gamma$:

$\Gamma' \vdash \alpha$ and $\Gamma' \vdash \neg\alpha$

$\Rightarrow \Gamma' \models \alpha$ and $\Gamma' \models \neg\alpha$ (by soundness)

$\Rightarrow \Gamma'$ not satisfiable: contradiction.

□