PART II:
PREDICATE CALCULUS

so far:

- *logic of the connectives* $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \ldots$ (as used in mathematics)
- *smallest unit:* propositions (nothing "inside")
- *deductive calculus:* checking logical validity and computing truth tables

$--\ >$ sound, complete, compact

now:

- go *more deeply into* the structure of propositions used in mathematics
- analyse grammatically correct use of *functions, relations, constants, variables* and *quantifiers*
- define *logical validity* in this refined language
- discover *axioms* and *rules of inference* (beyond those of propositional calculus) used in mathematical arguments
- prove: $--\ >$ sound, complete, compact

What is a mathematical structure?

A **group** G is a set G together with a binary operation $G^2 \ni (a, b) \mapsto ab \in G$ given by some binary function $f : G^2 \to G$ (the group operation) and a distinguished element e (the identity element).

So it is a triple $\langle G; f, e \rangle$.

A **field** has more/different structure it is

$\langle F; +, \times, 0, 1 \rangle$

We also want to consider **relations** like $<$.

The language of predicate calculus has symbols to represent these (functions, relations, constants) and enables suitable properties of the structures to be expressed.

8. The language of (first-order) predicate calculus

The language $\mathcal{L}^{\text{FOPC}}$ consists of the following symbols:

Logical symbols
connectives: $\rightarrow, \neg$
quantifier: $\forall$ ('for all')
variables: $x_0, x_1, x_2, \ldots$
3 punctuation marks: ( ) ,
equality symbol: $\doteq$

non-logical symbols:
predicate (or *relation*) *symbols:* $P_n^{(k)}$ for $n \geq 0, k \geq 1$ ($P_n^{(k)}$ is a k*-ary* predicate symbol)
function symbols: $f_n^{(k)}$ for $n \geq 0, k \geq 1$ ($f_n^{(k)}$ is a k*-ary* function symbol)
constant symbols: c_n for $n \geq 0$

8.1 Definition

(a) The **terms** of $\mathcal{L}^{\mathsf{FOPC}}$ are defined recursively as follows:
(i) Every variable is a term.
(ii) Every constant symbol is a term.
(iii) For each $n \geq 0, k \geq 1$, if $t_1, \ldots, t_k$ are terms, so is the string

$$f_n^{(k)}(t_1, \ldots, t_k)$$

(b) An **atomic formula** of $\mathcal{L}^{\mathsf{FOPC}}$ is any string of the form

$$P_n^{(k)}(t_1, \ldots, t_k) \text{ or } t_1 \doteq t_2$$

with $n \geq 0, k \geq 1$, and where all t_i are terms.

(c) The **formulas** of $\mathcal{L}^{\mathsf{FOPC}}$ are defined recursively as follows:
(i) Any atomic formula is a formula
(ii) If ϕ, ψ are formulas, then so are $\neg\phi$ and $(\phi \to \psi)$
(iii) If ϕ is a formula, then for any variable x_i so is $\forall x_i \phi$

8.2 Examples

c_0; c_3; x_5; $f_3^{(1)}(c_2)$; $f_4^{(2)}(x_1, f_3^{(1)}(c_2))$ are all terms

$f_2^{(3)}(x_1, x_2)$ is *not* a term (wrong arity)

$P_0^{(3)}(x_4, c_0, f_3^{(2)}(c_1, x_2))$ and $f_1^{(2)}(c_5, c_6) \doteq x_{11}$ are atomic formulas

$f_3^{(1)}(c_2)$ is a term, but no formula

$\forall x_1 f_2^{(2)}(x_1, c_7) \doteq x_2$ is a formula, not atomic

$\forall x_2 P_0^{(1)}(x_3)$ is a formula

8.3 Remark

We have **unique readability** for terms, for atomic formulas, and for formulas.

8.4 Interpretations and logical validity for $\mathcal{L}^{\mathsf{FOPC}}$ (Informal discussion)

(A) Consider the formula

$$\phi_1 : \ \forall x_1 \forall x_2 (x_1 \doteq x_2 \to f_5^{(1)}(x_1) \doteq f_5^{(1)}(x_2))$$

Given that $\doteq$ is to be interpreted as equality, $\forall$ as 'for all', and the $f_n^{(k)}$ as actual functions (in k arguments), ϕ_1 *should always be true*. We shall write

$$\models \phi_1$$

and say 'ϕ_1 is **logically valid**'.

(B) Consider the formula

$$\phi_2 : \ \forall x_1 \forall x_2 (f_7^{(2)}(x_1, x_2) \doteq f_7^{(2)}(x_2, x_1) \to x_1 \doteq x_2)$$

Then ϕ_2 may be false or true depending on the situation:

- If we interpret $f_7^{(2)}$ as $+$ on $\mathbb{N}$, then ϕ_2 is false, e.g. 1+2=2+1, but $1 \neq 2$. So in this interpretation, ϕ_2 is false and $\neg\phi_2$ is true. We will want to write

$$\langle \mathbb{N}; + \rangle \models \neg\phi_2$$

- If we interpret $f_7^{(2)}$ as minus (-) on $\mathbb{R}$, then ϕ_2 becomes true: if $x_1 - x_2 = x_2 - x_1$, then $2x_1 = 2x_2$, and hence $x_1 = x_2$.
So

$$\langle \mathbb{R}; - \rangle \models \phi_2$$

So we will need an appropriate formalism of interpreting our formulas and determining their validity.

8.5 Free and bound variables
(Informal discussion)

There is a further complication: Consider the formula

$$\phi_3 : \forall x_0 P_0^{(2)}(x_1, x_0)$$

Under the interpretation $\langle \mathbb{N}, \leq \rangle$ you cannot tell whether $\langle \mathbb{N}, \leq \rangle \models \phi_3$:

- if we put $x_1 = 0$ then yes
- if we put $x_1 = 2$ then no.

So it depends on the value we assign to x_1 (like in propositional calculus: truth value of $p_0 \wedge p_1$ depends on the valuation).

In ϕ_3 we **can** assign a value to x_1 because x_1 occurs **free** in ϕ_3.

For x_0, however, it makes no sense to assign a particular value; because x_0 is **bound** in ϕ_3 by the quantifier $\forall x_0$.