10. Free and bound variables

Recall Example 9.5: The formula

$$\phi = \forall x_0 \forall x_1 (f(x_0, x_2) \doteq f(x_1, x_2) \rightarrow x_0 \doteq x_1)$$

- is true in $\langle \mathbb{Z}; \cdot \rangle$ under any assignment v with $v(x_2) = 2$
- but false when $v(x_2) = 0$.

Whether or not $\mathcal{A} \models \phi[v]$ only depends on $v(x_2)$, not on $v(x_0)$ or $v(x_1)$.

The reason is: the variables x_0, x_1 are covered by a quantifier ($\forall$); we say they are "**bound**" (definition to follow!).

But the occurrence of x_2 is not "bound" by a quanitifer, but rather is "**free**".

10.1 Definition

Let $\mathcal{L}$ be a first-order language, ϕ an $\mathcal{L}$-formula, and $x \in \{x_0, x_1, \ldots\}$ a variable occurring in ϕ.

An occurrence of x in ϕ is **free**, if
(i) ϕ is atomic, or
(ii) $\phi = \neg\psi$ resp. $\phi = (\chi \rightarrow \rho)$ and x occurs free in ψ resp. in χ or ρ, or
(iii) $\phi = \forall x_i \psi$, x occurs free in ψ, and $x \neq x_i$.

Every other occurrence of x in ϕ is called **bound**.

In particular, if $x = x_i$ and $\phi = \forall x_i \psi$, then x is bound in ϕ.

10.2 Example

$$(\exists x_0 P(\underbrace{x_0}_{b}, \underbrace{x_1}_{f}) \vee \forall x_1 (P(\underbrace{x_0}_{f}, \underbrace{x_1}_{b}) \rightarrow \exists x_0 P(\underbrace{x_0}_{b}, \underbrace{x_1}_{b})))$$

(This page intentionally left (almost) blank)

Lecture 10 - 2.01/12

10.3 Lemma
Let $\mathcal{L}$ be a language, let $\mathcal{A}$ be an $\mathcal{L}$-structure, let v, v' be assignments in $\mathcal{A}$ and let ϕ be an $\mathcal{L}$-formula.

Suppose $v(x_i) = v'(x_i)$ for every variable x_i with a free *occurrence in ϕ.*

Then

$$\mathcal{A} \models \phi[v] \text{ iff } \mathcal{A} \models \phi[v'].$$

Proof:
For ϕ atomic: exercise

Now use induction on the length of ϕ:
- $\phi = \neg\psi$ and $\phi = (\chi \to \rho)$: easy
- $\phi = \forall x_i \psi$:

IH: Assume the Lemma holds for ψ.

Let
Free (ϕ):=$\{x_j \mid x_j$ occurs free in $\phi\}$
Free (ψ):=$\{x_j \mid x_j$ occurs free in $\psi\}$

$\Rightarrow x_i \notin \mathsf{Free}(\phi)$ and

$$\mathsf{Free}(\phi) = \mathsf{Free}(\psi) \setminus \{x_i\}$$

Assume $\mathcal{A} \models \forall x_i \psi[v]$ $(\star)$

to show: for any $v^\star$ agreeing with v' except possibly at x_i: $\mathcal{A} \models \psi[v^\star]$.

for all $x_j \in \mathsf{Free}(\phi)$:

$$v^\star(x_j) = v(x_j) = v'(x_j).$$

Let $v^+(x_j) := \begin{cases} v(x_j) & \text{if } j \neq i \\ v^\star(x_j) & \text{if } j = i \end{cases}$

Then v^+ agrees with v except possibly at x_i.

Hence, by $(\star)$, $\mathcal{A} \models \psi[v^+]$.

But $v^\star(x_j) = v^+(x_j)$ for all $x_j \in \mathsf{Free}(\psi)$.

$\Rightarrow$ by IH, $\mathcal{A} \models \psi[v^\star]$ $\square$

10.4 Corollary
Let $\mathcal{L}$ be a language, $\alpha, \beta \in$ Form($\mathcal{L}$). Assume the variable x_i has no free occurrence in α. Then

$$\models (\forall x_i(\alpha \to \beta) \to (\alpha \to \forall x_i \beta)).$$

Proof:
Let $\mathcal{A}$ be an $\mathcal{L}$-structure and let v be an assignment in $\mathcal{A}$ such that
$\mathcal{A} \models \forall x_i(\alpha \to \beta)[v]$ $(\star)$

to show: $\mathcal{A} \models (\alpha \to \forall x_i \beta)[v]$.

So suppose $\mathcal{A} \models \alpha[v]$
to show: $\mathcal{A} \models \forall x_i \beta[v]$.

So let $v^\star$ be an assignment agreeing with v except possibly at x_i.
We want: $\mathcal{A} \models \beta[v^\star]$

x_i is *not* free in α $\Rightarrow_{10.3}$ $\mathcal{A} \models \alpha[v^\star]$
$(\star) \Rightarrow \mathcal{A} \models (\alpha \to \beta)[v^\star]$
$\Rightarrow \mathcal{A} \models \beta[v^\star]$ $\square$

10.5 Definition
A formula ϕ without free (occurrence of) variables is called a **statement** or a **sentence**.

If ϕ is a sentence then, for any $\mathcal{L}$-structure $\mathcal{A}$, whether or not $\mathcal{A} \models \phi[v]$ does not depend on the assignment v.

So we write $\mathcal{A} \models \phi$ if $\mathcal{A} \models \phi[v]$ for some/all v.

Say: ϕ is **true** in $\mathcal{A}$, or $\mathcal{A}$ is a **model** of ϕ.

($\rightsquigarrow$ 'Model Theory')

10.6 Example

Let $\mathcal{L} = \{f, c\}$ be a language, where f is a binary function symbol, and c is a constant symbol.

Consider the sentences (we write x, y, z instead of x_0, x_1, x_2)

$$\begin{array}{ll} \phi_1: & \forall x \forall y \forall z f(x, f(y, z)) \doteq f(f(x, y), z) \\ \phi_2: & \forall x \exists y (f(x, y) \doteq c \wedge f(y, x) \doteq c) \\ \phi_3: & \forall x (f(x, c) \doteq x \wedge f(c, x) \doteq x) \end{array}$$

and let $\phi = \phi_1 \wedge \phi_2 \wedge \phi_3$.

Let $\mathcal{A} = \langle A; \circ; e \rangle$ be an $\mathcal{L}$-structure (i.e. $\circ$ is an interpretation of f, and e is an interpretation of c.)

Then $\mathcal{A} \models \phi$ iff $\mathcal{A}$ is a group.

10.7 Example

Let $\mathcal{L} = \{E\}$ be a language with $E = P_i^{(2)}$ a binary relation symbol. Consider

$$\begin{aligned}
\chi_1 &: \ \forall x E(x,x) \\
\chi_2 &: \ \forall x \forall y (E(x,y) \leftrightarrow E(y,x)) \\
\chi_3 &: \ \forall x \forall y \forall z (E(x,y) \rightarrow (E(y,z) \rightarrow E(x,z)))
\end{aligned}$$

Then for any $\mathcal{L}$-structure $\langle A; R\rangle$:
$\langle A; R\rangle \models (\chi_1 \wedge \chi_2 \wedge \chi_3)$ iff
R is an equivalence relation on A.

Note: Most mathematical concepts can be captured by first-order formulas.

10.8 Example

Let P be a 2-place (i.e. binary) predicate symbol, $\mathcal{L} := \{P\}$. Consider the statements

ψ_1 : $\forall x \forall y (P(x,y) \dot{\vee}\ x \doteq y \dot{\vee}\ P(y,x))$
($\dot{\vee}$ means *either - or exclusively:*
$(\alpha \dot{\vee}\ \beta) :\Leftrightarrow ((\alpha \vee \beta) \wedge \neg(\alpha \wedge \beta)))$

ψ_2 : $\forall x \forall y \forall z ((P(x,y) \wedge P(y,z)) \rightarrow P(x,z))$

ψ_3 : $\forall x \forall z (P(x,z) \rightarrow \exists y (P(x,y) \wedge P(y,z)))$

ψ_4 : $\forall y \exists x \exists z (P(x,y) \wedge P(y,z))$

These are the axioms for a **dense linear order without endpoints**. Let $\psi = (\psi_1 \wedge \ldots \wedge \psi_4)$. Then $\langle \mathbb{Q}; < \rangle \models \psi$ and $\langle \mathbb{R}; < \rangle \models \psi$.

However: *The* **Dedekind Completeness** *of* $\langle \mathbb{R}; < \rangle$ *is* **not** *captured in 1st-order terms using the langauge* $\mathcal{L}$*, but rather in 2nd-order terms, where also quantification over* subsets, *rather than only over* elements *of* $\mathbb{R}$ *is used:*

$$\forall A, B \subseteq \mathbb{R} ((A \ll B) \rightarrow \exists c \in \mathbb{R} (A \lll \{c\} \lll B),$$

where $A \ll B$ *means that* $a < b$ *for every* $a \in A$ *and every* $b \in B$ *etc. We will see it* **cannot** *be captured in first order terms.*

10.9 Example: ACF_0: *Algebraically closed fields of characteristic zero.*

$\mathcal{L} := \{+, \times, 0, 1\}$, *language of rings*

Commutative, associative, distributive laws; the existence of multiplicative inverse of non-zero elements;

Characteristic 0: $1 + 1 \neq 0, 1 + 1 + 1 \neq 0, \ldots$

For each $n = 2, 3, 4, \ldots$ *a sentence* ψ_n *asserting that every non-constant polynomial has a root. (This is automatic for* $n = 1$*).*

$$\forall a_0 \ldots \forall a_n [\neg a_n = 0 \to \exists x (a_n x^n + \ldots + a_0 = 0)]$$

This set of axioms is **complete** *and* **decidable**. *(Complete: every sentence* ϕ*, either* ϕ *or* $\neg\phi$ *is a logical consequence of the axioms.)*

Examples 10.7, 10.8, 10.9 are of the type which will be explored in Part C Model Theory.

10.10 Example: Peano Arithmetic (PA)

This is historically a very important system, studied in Part C Godel's Incompleteness Thms. It is not complete and not decidable.

$\mathcal{L} := \{0, +, \times, s\}$, $\mathcal{A} = \langle \mathbb{N}; 0, +, \times, s : n \mapsto n+1 \rangle$

The unary function s is called the "successor function". Suitable sentences express: it is injective and its range is everything except 0.

Suitable sentences give axioms for $+, \times$.

Induction: for every unary formula ϕ the axiom

$$[\phi(0) \wedge \forall x(\phi(x) \to \phi(s(x)))] \to \forall y \phi(y)$$

This is weaker than the second order system proposed by Peano which states induction for every **subset** *of $\mathbb{N}$.*

10.11 Example: Set Theory

Several ways of axiomatizing a system for Set Theory, in which all (?) mathematics can be carried out.

The most popular system ZFC is introduced in B1.2 Set Theory, and more formally in Part C Axiomatic Set Theory. ZFC has:

$\mathcal{L} := \{\in\}$, *a binary relation for set membership*

Axioms: existence of empty set, pairs, unions, power set,.....

10.12 Example: Second order logic

Lose completeness, compactness.