13. The Completeness Theorem for Predicate Calculus

13.1 Theorem (Gödel)
Let $\Gamma \subseteq$ *Form*$(\mathcal{L})$, $\phi \in$ *Form*$(\mathcal{L})$.

$$\text{If } \Gamma \models \phi \text{ then } \Gamma \vdash \phi.$$

Two additional assumptions:

- Assume all $\gamma \in \Gamma$ and ϕ are *sentences* – the Theorem is true more generally, but the proof is much harder and applications are typically to sentences.

- Further assumption (for the start – later we do the general case): *no* $\doteq$*-symbol in any formula of* Γ *or in* ϕ.

First Step
Call $\Delta \subseteq \mathrm{Sent}(\mathcal{L})$ **consistent** if for no sentence ψ, both $\Delta \vdash \psi$ and $\Delta \vdash \neg\psi$.

13.2. To prove 13.1 it is enough to prove:
($\star$) Every consistent set of sentences has a model.

i.e. Δ consistent $\Rightarrow$
there is an $\mathcal{L}$-structure $\mathcal{A}$ such that
$\mathcal{A} \models \delta$ for every $\delta \in \Delta$.

Proof of 13.2: Assume $\Gamma \models \phi$ and assume ($\star$).
$\Rightarrow \Gamma \cup \{\neg\phi\}$ has no model
$\Rightarrow_{(\star)} \Gamma \cup \{\neg\phi\}$ is not consistent
$\Rightarrow \Gamma \cup \{\neg\phi\} \vdash \psi$ and $\Gamma \cup \{\neg\phi\} \vdash \neg\psi$ for some ψ
$\Rightarrow_{\mathrm{DT}}$ $\Gamma \vdash (\neg\phi \to \psi)$ and $\Gamma \vdash (\neg\phi \to \neg\psi)$ for some ψ
But $\Gamma \vdash ((\neg\phi \to \psi) \to ((\neg\phi \to \neg\psi) \to \phi))$ [taut.]
$\Rightarrow \Gamma \vdash \phi$ [2xMP] $\square_{13.2}$

Second Step

We shall need an *infinite* supply of constant symbols.

To do this, let ϕ' be the formula obtained by replacing every occurrence of c_n by c_{2n}.

For $\Delta \subseteq \mathrm{Form}(\mathcal{L})$ let

$$\Delta' := \{\phi' \mid \phi \in \Delta\}$$

Then

13.3 Lemma

(a) Δ consistent $\Rightarrow$ Δ' consistent

(b) Δ' has a model $\Rightarrow$ Δ has a model.

Proof: Easy exercise. $\square$

Third Step

- $\Delta \subseteq$ Sent($\mathcal{L}$) is called **maximal consistent** if Δ is consistent, and for any $\psi \in$ Sent($\mathcal{L}$): $\Delta \vdash \psi$ or $\Delta \vdash \neg\psi$.

- $\Delta \subseteq$ Sent($\mathcal{L}$) is called **witnessing** if for all $\psi \in$ Form($\mathcal{L}$) with Free(ψ) $\subseteq \{x_i\}$ and with $\Delta \vdash \exists x_i \psi$ there is some $c_j \in$ Const($\mathcal{L}$) such that $\Delta \vdash \psi[c_j/x_i]$

13.4 To prove CT it is enough to show:
Every maximal consistent witnessing set Δ of sentences has a model.

For the proof of 13.4 we need 2 Lemmas:

13.5 Lemma
If $\Delta \subseteq$ *Sent*$(\mathcal{L})$ *is consistent, then for any sentence* ψ*, either* $\Delta \cup \{\psi\}$ *or* $\Delta \cup \{\neg\psi\}$ *is consistent.*

Proof: Exercise – as for Propositional Calculus. $\square$.

13.6 Lemma
Assume $\Delta \subseteq$ *Sent*$(\mathcal{L})$ *is consistent,* $\exists x_i \psi \in$ *Sent*$(\mathcal{L})$*,* $\Delta \vdash \exists x_i \psi$*, and* c_j *is* not *occurring in* ψ *nor in any* $\delta \in \Delta$*.*

Then $\Delta \cup \{\psi[c_j/x_i]\}$ *is consistent.*

Proof:
Assume, for a contradiction, that there is some $\chi \in \text{Sent}(\mathcal{L})$ such that

$$\Delta \cup \{\psi[c_j/x_i]\} \vdash \chi \text{ and } \Delta \cup \{\psi[c_j/x_i]\} \vdash \neg\chi.$$

May assume that c_j does *not* occur in χ (since $\vdash (\chi \to (\neg\chi \to \theta))$ for *any* sentence θ).

By DT, $\Delta \vdash (\psi[c_j/x_i] \to \chi)$
and $\Delta \vdash (\psi[c_j/x_i] \to \neg\chi)$.

Then also

$$\Delta \vdash (\psi \to \chi) \text{ and } \Delta \vdash (\psi \to \neg\chi)$$

(Exercise Sheet # 4 (2)(ii))

By $\forall$, $\Delta \vdash \forall x_i(\psi \to \chi)$
and $\Delta \vdash \forall x_i(\psi \to \neg\chi)$
(note that $x_i \notin$ Free(δ) for any $\delta \in \Delta \subseteq$ Sent($\mathcal{L}$)).

Now: $\vdash (\forall x_i(A \to B) \to (\exists x_i A \to B))$
for any $A, B \in$ *Form*($\mathcal{L}$) *with* $x_i \notin$ *Free*(B)
(Exercise Sheet $\sharp$ 4, (2)(i))

MP $\Rightarrow$ $\Delta \vdash (\exists x_i \psi \to \chi)$
and $\Delta \vdash (\exists x_i \psi \to \neg\chi)$
($\chi, \neg\chi \in$ Sent($\mathcal{L}$), so $x_i \notin$ Free(χ))

By hypothesis, $\Delta \vdash \exists x_i \psi$
$\Rightarrow$ by MP, $\Delta \vdash \chi$ and $\Delta \vdash \neg\chi$
contradicting consistency of Δ.

$\square_{13.6}$

Proof of 13.4:
Let Δ be any consistent set of sentences.

to show: Δ has a model *assuming that any maximal consistent, witnessing set of sentences has a model*.

By 13.3(a), Δ' is consistent and does not contain any c_{2m+1}.

Let $\phi_1, \phi_2, \phi_3, \ldots$ be an enumeration of $\text{Sent}(\mathcal{L}' \cup \{c_1, c_3, c_5, \ldots\})$.

Construct finite sets $\subseteq \text{Sent}(\mathcal{L}' \cup \{c_1, c_3, c_5, \ldots\})$

$$\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \ldots$$

such that $\Delta' \cup \Gamma_n$ is consistent for each $n \geq 0$ as follows:

Let $\Gamma_0 := \emptyset$.

If Γ_n has been constructed let

$$\Gamma_{n+1/2} := \begin{cases} \Gamma_n \cup \{\phi_{n+1}\} & \text{if } \Delta' \cup \Gamma_n \cup \{\phi_{n+1}\} \\ & \quad \text{is consistent} \\ \Gamma_n \cup \{\neg\phi_{n+1}\} & \text{otherwise} \end{cases}$$

$\Rightarrow \Gamma_{n+1/2}$ is consistent (Lemma 13.5)

Now, if $\neg\phi_{n+1} \in \Gamma_{n+1/2}$ or if ϕ_{n+1} is *not* of the form $\exists x_i \psi$, let $\Gamma_{n+1} := \Gamma_{n+1/2}$.

If not, i.e. if $\phi_{n+1} = \exists x_i \psi \in \Gamma_{n+1/2}$ then $\Delta' \cup \Gamma_{n+1/2} \vdash \exists x_i \psi$.

Choose m large enough such that c_{2m+1} does not occur in any formula in $\Delta' \cup \Gamma_{n+1/2} \cup \{\psi\}$ (possible since $\Gamma_{n+1/2} \cup \{\psi\}$ is finite and Δ' has only even constants).

Let $\Gamma_{n+1} := \Gamma_{n+1/2} \cup \{\psi[c_{2m+1}/x_i]\}$
$\Rightarrow$ by Lemma 13.6, Γ_{n+1} is consistent.

Let $\Gamma := \Delta' \cup \bigcup_{n\geq 0} \Gamma_n$.

$\Rightarrow$ Γ is maximal consistent
(as in Propositional Calculus)
and Γ is witnessing (by construction).

By assumption, Γ has a model, say $\mathcal{A}$.

$\Rightarrow$ in particular, $\Gamma \models \delta$ for any $\delta \in \Delta'$

$\Rightarrow$ by Lemma 13.3(b), Δ has a model

$\square_{13.4}$

So to prove CT it remains to show:
Every maximal consistent witnessing set Δ of sentences has a model.