13.7 Theorem (CT after reduction 13.4)
Let Γ be a maximal consistent witnessing set of sentences not containing a $\doteq$-symbol. Then Γ has a model.

Proof:
Let $A := \{t \in \text{Term}(\mathcal{L}) \mid t \text{ is closed}\}$
(recall: t **closed** means no variables in t).

A will be the domain of our model $\mathcal{A}$ of Γ
($\mathcal{A}$ is called **term model**).

For $P = P_n^{(k)} \in \text{Pred}(\mathcal{L})$ resp. $f = f_n^{(k)} \in \text{Fct}(\mathcal{L})$ resp. $c = c_n \in \text{Const}(\mathcal{L})$ define the interpretations $P_{\mathcal{A}}$ resp. $f_{\mathcal{A}}$ resp. $c_{\mathcal{A}}$ by

$$\begin{array}{rcl} P_{\mathcal{A}}(t_1,\ldots,t_k) \text{ holds} & :\Leftrightarrow & \Gamma \vdash P(t_1,\ldots,t_k) \\ f_{\mathcal{A}}(t_1,\ldots,t_k) & := & f(t_1,\ldots,t_k) \\ c_{\mathcal{A}} & := & c \end{array}$$

The claim is: $\mathcal{A} \models \Gamma$

to show: $\mathcal{A} \models \Gamma$
(i.e. $\mathcal{A} \models \Gamma[v]$ for some/all assignments v in $\mathcal{A}$: note that Γ contains only sentences).

Let v be an assignment in $\mathcal{A}$,
say $v(x_i) =: s_i \in A$ for $i = 0, 1, 2, \ldots$.

Claim 1: For any $u \in \mathrm{Term}(\mathcal{L})$: $\widetilde{v}(u) = u[\vec{s}/\vec{x}]$
(:= the closed term obtained by replacing each x_i in u by s_i)

Proof: by induction on u
- $u = x_i \Rightarrow$
$\widetilde{v}(u) = v(x_i) = s_i = x_i[s_i/x_i] = u[\vec{s}/\vec{x}]$
- $u = c \in \mathrm{Const}(\mathcal{L}) \Rightarrow$
$\widetilde{v}(u[\vec{s}/\vec{x}]) = \widetilde{v}(u) = v(c) = c_{\mathcal{A}}$
- $u = f(t_1, \ldots, t_k) \Rightarrow$

$$
\begin{array}{rll}
\widetilde{v}(u) & := f_{\mathcal{A}}(\widetilde{v}(t_1), \ldots, \widetilde{v}(t_k)) & \\
& = f_{\mathcal{A}}(t_1[\vec{s}/\vec{x}], \ldots, t_k[\vec{s}/\vec{x}]) & \text{by IH} \\
& = f(t_1[\vec{s}/\vec{x}], \ldots, t_k[\vec{s}/\vec{x}]) & \text{by def. of} f_{\mathcal{A}} \\
& = f(t_1, \ldots, t_k)[\vec{s}/\vec{x}] & \text{by def. of subst.} \\
& = u[\vec{s}/\vec{x}] & \Box_{\text{Claim 1}}
\end{array}
$$

Claim 2: *For any $\phi \in$ Form($\mathcal{L}$) without $\doteq$-symbol:*

$$\mathcal{A} \models \phi[v] \text{ iff } \Gamma \vdash \phi[\vec{s}/\vec{x}],$$

where $\phi[\vec{s}/\vec{x}]$:= the sentence obtained by replacing each *free* occurrence of x_i by s_i: note that s_i *is* free for x_i in ϕ because s_i is a *closed* term.

Proof: by induction on ϕ

ϕ **atomic**, i.e.
$\phi = P(t_1, \ldots, t_k)$ for some $P = P_n^{(k)} \in$ Pred($\mathcal{L}$)

Then

$$\begin{array}{lll} & \mathcal{A} \models \phi[v] & \\ \text{iff} & P_{\mathcal{A}}(\widetilde{v}(t_1), \ldots, \widetilde{v}(t_k)) & \text{[def. of `}\models\text{']} \\ \text{iff} & P_{\mathcal{A}}(t_1[\vec{s}/\vec{x}], \ldots, t_k[\vec{s}/\vec{x}]) & \text{[Claim 1]} \\ \text{iff} & \Gamma \vdash P(t_1[\vec{s}/\vec{x}], \ldots, t_k[\vec{s}/\vec{x}]) & \text{[def. of } P_{\mathcal{A}}\text{]} \\ \text{iff} & \Gamma \vdash P(t_1, \ldots, t_k)[\vec{s}/\vec{x}] & \text{[def. subst.]} \\ \text{iff} & \Gamma \vdash \phi[\vec{s}/\vec{x}] & \end{array}$$

Note that Claim 2 might be false for formulas of the form $t_1 \doteq t_2$: might have $\Gamma \vdash c_0 \doteq c_1$, but c_0, c_1 are distinct elements in A.

Induction Step

$$
\begin{array}{lll}
 & \mathcal{A} \models \neg\phi[v] & \\
\text{iff} & \text{not } \mathcal{A} \models \phi[v] & [\text{def. of `}\models\text{'}] \\
\text{iff} & \text{not } \Gamma \vdash \phi[\vec{s}/\vec{x}] & [\text{IH}] \\
\text{iff} & \Gamma \vdash \neg\phi[\vec{s}/\vec{x}] & [\Gamma \text{ max. cons.}]
\end{array}
$$

$$
\begin{array}{lll}
 & \mathcal{A} \models (\phi \to \psi)[v] & \\
\text{iff} & \text{not } \mathcal{A} \models \phi[v] \text{ or } \mathcal{A} \models \psi[v] & [\text{def. `}\models\text{'}] \\
\text{iff} & \text{not } \Gamma \vdash \phi[\vec{s}/\vec{x}] \text{ or } \Gamma \vdash \psi[\vec{s}/\vec{x}] & [\text{IH}] \\
\text{iff} & \Gamma \vdash \neg\phi[\vec{s}/\vec{x}] \text{ or } \Gamma \vdash \psi[\vec{s}/\vec{x}] & [\Gamma \text{ max.}] \\
\text{iff} & \Gamma \vdash (\neg\phi[\vec{s}/\vec{x}] \vee \psi[\vec{s}/\vec{x}]) & [\text{def. `}\vdash\text{'}] \\
\text{iff} & \Gamma \vdash (\phi[\vec{s}/\vec{x}] \to \psi[\vec{s}/\vec{x}]) & [\text{taut.}] \\
\text{iff} & \Gamma \vdash (\phi \to \psi)[\vec{s}/\vec{x}] & [\text{def. subst.}]
\end{array}
$$

$\forall$-step '$\Rightarrow$'

Suppose $\mathcal{A} \models \forall x_i \phi[v]$ $(\star)$

but not $\Gamma \vdash (\forall x_i \phi)[\vec{s}/\vec{x}]$

$\Rightarrow \Gamma \vdash (\neg\forall x_i \phi)[\vec{s}/\vec{x}]$ (Γ max.)

$\Rightarrow \Gamma \vdash (\exists x_i \neg\phi)[\vec{s}/\vec{x}]$ (Exercise)

Now let ϕ' be the result of substituting each free occurrence of x_j in ϕ by s_j for all $j \neq i$.

$\Rightarrow (\exists x_i \neg\phi)[\vec{s}/\vec{x}] = \exists x_i \neg\phi'$
$\Rightarrow \Gamma \vdash \exists x_i \neg\phi'$

Γ witnessing $\Rightarrow$
$\Gamma \vdash \neg\phi'[c/x_i]$ for some $c \in \mathrm{Const}(\mathcal{L})$

Define

$$v^\star(x_j) := \begin{cases} v(x_j) & \text{if } j \neq i \\ c & \text{if } j = i \end{cases} \quad \text{and } s_j^\star := \begin{cases} s_j & \text{if } j \neq i \\ c & \text{if } j = i \end{cases}$$

$\Rightarrow \neg\phi'[c/x_i] = \neg\phi[\vec{s^\star}/\vec{x}]$
$\Rightarrow \Gamma \vdash \neg\phi[\vec{s^\star}/\vec{x}]$
$\Rightarrow \mathcal{A} \models \neg\phi[v^\star]$ [IH]

But, by $(\star)$, $\mathcal{A} \models \phi[v^\star]$: contradiction.

[**Note:** There was a typo above when I made the video: "$\Rightarrow \mathcal{A} \models \neg\phi[v^\star]$" just above had been "$\Rightarrow \Gamma \models \neg\phi[v^\star]$".]

$\forall$-**step '$\Leftarrow$':**
Suppose $\mathcal{A} \not\models \forall x_i \phi[v]$
$\Rightarrow$ for some $v^\star$ agreeing with v except possibly at x_i

$$\mathcal{A} \models \neg\phi[v^\star]$$

Let $s^\star_j := \begin{cases} s_j & \text{for } j \neq i \\ v^\star(x_j) & \text{for } j = i \end{cases}$

IH $\Rightarrow$ $\Gamma \vdash \neg\phi[\vec{s^\star}/\vec{x}]$,
i.e. $\Gamma \vdash \neg\phi'[s^\star_i/x_i]$,
where ϕ' is the result of substituting each free occurrence of x_j in ϕ by s_j for all $j \neq i$

$\Rightarrow \Gamma \vdash \exists x_i \neg\phi'$

(**Exercise:**
$\chi \in$ Form($\mathcal{L}$), Free(χ) $\subseteq \{x_i\}$, s a closed term
$\Rightarrow$ $\vdash (\chi[s/x_i] \to \exists x_i \chi))$

So

$$
\begin{aligned}
& \Gamma \vdash \neg\forall x_i \neg\neg\phi' \\
\Rightarrow\ & \Gamma \vdash \neg\forall x_i \phi' \\
\Rightarrow\ & \Gamma \vdash (\neg\forall x_i \phi)[\vec{s}/\vec{x}] \\
\Rightarrow\ & \Gamma \nvdash (\forall x_i \phi)[\vec{s}/\vec{x}]
\end{aligned}
$$

$\square_{\text{Claim 2}}$

Now choose any $\phi \in \Gamma \subseteq \text{Sent}(\mathcal{L})$

$\Rightarrow\ \phi[\vec{s}/\vec{x}] = \phi$

$\Rightarrow\ \mathcal{A} \models \phi[v]$, i.e. $\mathcal{A} \models \phi$ [Claim 2]

$\Rightarrow\ \mathcal{A} \models \Gamma$

$\square_{13.7}$

13.8 Modification required for $\doteq$–symbol

Define an equivalence relation E on A by

$$t_1 E t_2 \text{ iff } \Gamma \vdash t_1 \doteq t_2$$

(easy to check: this *is* an equivalence relation, e.g. transitivity = (1)(ii) of sheet $\sharp$ 4).

Let A/E be the set of equivalence classes t/E (with $t \in A$).

Define $\mathcal{L}$-structure $\mathcal{A}/E$ with domain A/E by

$$\begin{array}{rcl} P_{\mathcal{A}/E}(t_1/E, \ldots, t_k/E) & :\Leftrightarrow & \Gamma \vdash P(t_1, \ldots, t_k) \\ f_{\mathcal{A}/E}(t_1/E, \ldots, t_k/E) & := & f_{\mathcal{A}}(t_1, \ldots, t_k) \\ c_{\mathcal{A}/E} & := & c_{\mathcal{A}}/E \end{array}$$

check: independence of representatitves of t/E (this is the purpose of Axiom **A7**).

Rest of the proof is much the same as before.

$\square_{13.1}$