# 15. Another example: Algebraically closed fields

**15.1** Let $\mathcal{L}_r = \{+, \times, 0, 1\}$ be the language of rings. In this language one can formulate axioms that say that a given $\mathcal{L}_r$ structure is a ring, or that it is a field:

$$\begin{array}{ll} \psi_1 : & \forall x \forall y x + y = y + x \\ \psi_2 : & \forall x \forall y \forall z (x + y) + z = x + (y + z) \\ \psi_3 : & etc \end{array}$$

**15.2 There are many examples of fields:** $\langle A; +, \times, 0, 1\rangle$ where $A$ is: $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, and also **finite fields** $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, where $p$ is a prime number, and their extensions (see later).

These are not elementarily equivalent: e.g. $\mathbb{C}$ has a solution to $x^2 = -1$ but $\mathbb{R}$ doesn't. Both solve $x^2 = 2$ but $\mathbb{Q}$ doesn't, while a finite field has $\mathbb{F}_p$ **characteristic** $p$, i.e. $p = 0$ while $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ have characteristic 0 (Definition: no positive integer $n = 0$).

**15.3** A field $K$ contains a **prime field**, smallest subfield, which is either $\mathbb{F}_p$ if char$(K) = p > 0$ or $\mathbb{Q}$ if char$(K) = 0$.

**15.4** One can **extend** fields algebraically, like $\mathbb{Q}(\sqrt{2})$ extends $\mathbb{Q}$, or $\mathbb{C} = \mathbb{R}(i)$ extends $\mathbb{R}$, or by **transcendental elements**, like the field of rational functions $\mathbb{Q}(x)$ extends $\mathbb{Q}$.

**15.5** If $K \subset L$ are fields, a **transcendence basis** for $L$ over $K$ is a set $X \subset L$ of algebraically independent elements such that $L$ is algebraic over $K(X)$. Any two of these have the same **cardinality** (that's a theorem), called the **transcendence degree** of the extension $L/K$.

**15.6. Examples.**
tr.deg.$(\mathbb{Q}(\sqrt{2})/\mathbb{Q}) = 0$,
tr.deg.$(\mathbb{C}/\mathbb{R}) = 0$,
tr.deg.$(\mathbb{Q}(x)/\mathbb{Q}) = 1$,
tr.deg.$(\mathbb{C}/\mathbb{Q}) = |\mathbb{C}| = 2^{\aleph_0}$.

**15.7 Theorem (Steinitz)**. *Two algebraically closed fields are isomorphic if they have same prime field and same transcendence degree over the prime field.*

**"Proof".** By back and forth construction of an ismorphism starting with transcendence bases $X_1, X_2$ of each field $K_1, K_2$ over prime field $F$. $F(X_1), F(X_2)$ are isomorphic under any bijection $X_1 \to X_2$. Further elements are algebraic. Taking an element of $K_1$ (say) we can solve the "same" equation in $K_2$.

Thus we see that any algebraically closed field of characteristic 0 and transcendence degree over $\mathbb{Q}$ equal to $2^{\aleph_0}$ is **isomorphic** to $\mathbb{C}$.

**15.8.** We can axiomatize the theory of algebraically closed fields of characteristic $p \geq 0$ (recall 10.9). Characteristic 0 is axiomatized by the sequence of formulas

$1 + \ldots + 1 \neq 0$ (with $n$ 1's, $n = 2, 3, \ldots$)

Algebraic closedness can be axiomatized by a sequence of axioms (for each degree $d = 2, \ldots$) saying that every monic polynomial of degree $d$ has a root:

$$\forall a_0 \ldots \forall a_{d-1} \exists x a_0 + a_1 x + \ldots a_{d-1} x^{d-1} + x^d = 0$$

We get **Theories** $\mathsf{ACF}_0$, $\mathsf{ACF}_p$.

**15.9.** By (a stronger version of) Löwenheim-Skolem, $\mathsf{ACF}_0$ is **complete** (max consistent) and is the theory of $\mathsf{Th}(\langle \mathbb{C}, +, \times, 0, 1 \rangle)$. Likewise $\mathsf{ACF}_p$ is complete.

This means that a theorem about $\mathbb{C}$ that can be expressed in $\mathcal{L}_r$ holds in any algebraically closed field of char 0, and has a proof in FOPC, even if it was proved using methods of analysis.

**15.10.** We can also prove things about $\mathbb{C}$ by going to positive characteristic! (i.e. $p > 0$)

Any proof in $\mathrm{ACF}_0$ can only use finitely many of the axioms $1 + \ldots + 1 \neq 0$ and hence goes through in any algebraically closed field whose characteristic is sufficiently large.

**15.11. Theorem.** *Let $\phi$ be a sentence in $\mathcal{L}_r$. TFAE:*

* $\mathbb{C} \models \phi$
* $\phi$ *holds in* **all** *alg closed fields of char 0*
* ... **some** ...
* *For arbitrarily large $p$, $\phi$ holds in some* $\mathrm{ACF}_p$
* $\phi$ *holds in any* $\mathrm{ACF}_p$ *provided $p > N(\phi)$*

One rather surprising application:

**15.12. Theorem** (Ax-Grothendieck) *Consider a polynomial map $F : \mathbb{C}^n \to \mathbb{C}^n$. If $F$ is injective then it is surjective.*

**"Proof"** A finite field $\mathbb{F}_p$ has a unique extension of each finite degree $n$, denoted $\mathbb{F}_q, q = p^n$. Likewise $\mathbb{F}_q, q = p^n$, and the algebraic closure $\overline{\mathbb{F}}$ of $\mathbb{F}_p$ is the union of these. If we consider a polynomial map $F : \overline{\mathbb{F}}^n \to \overline{\mathbb{F}}^n$ then the coefficients will be in $\mathbb{F}_q$ for some (large enough) $q$, and then $F$ maps $\mathbb{F}_q^n \to \mathbb{F}_q^n$ for this field and all its extensions. These fields are **finite**, so if a map is injective it must be surjective (and *vice versa*). So the statement holds in positive characteristic. But it is expressible in $\mathcal{L}_r$: for each $n, d$, with $F$ a "general" degree $d$ poly map:

$\forall$ coefficients(F injective $\to$F surjective)

So it holds in (is a theorem of) ACF$_0$.

# 16. Normal Forms

## (a) Prenex Normal Form

A formula is in **prenex normal form (PNF)** if it has the form

$$Q_1 x_{i_1} Q_2 x_{i_2} \cdots Q_r x_{i_r}\, \psi,$$

where each $Q_i$ is a quantifier
(i.e. either $\forall$ or $\exists$), and where
$\psi$ is a formula containing no quantifiers.

**16.1 PNF-Theorem**
*Every $\phi \in$ Form($\mathcal{L}$) is logically equivalent to an $\mathcal{L}$-formula in* **PNF***.*

*Proof:* Induction on $\phi$
(working in the language with $\forall, \exists, \neg, \wedge$):

$\phi$ atomic: OK

$\phi = \neg\psi$,
say $\phi \leftrightarrow \neg Q_1 x_{i_1} Q_2 x_{i_2} \cdots Q_r x_{i_r}\, \chi$

Then $\phi \leftrightarrow Q_1^- x_{i_1} Q_2^- x_{i_2} \cdots Q_r^- x_{i_r}\, \neg\chi$,
where $Q^- = \exists$ if $Q = \forall$, and $Q^- = \forall$ if $Q = \exists$

$\phi = (\chi \wedge \rho)$ with $\chi, \rho$ in PNF
Note that $\vdash (\forall x_j \psi[x_j/x_i] \leftrightarrow \forall x_i \psi)$,
provided $x_j$ does not occur in $\psi$ (Ex. 12.5)

So w.l.o.g. the variables quantified over in $\chi$ do not occur in $\rho$ and vice versa.

But then, e.g. $(\forall x \alpha \wedge \exists y \beta) \leftrightarrow \forall x \exists y (\alpha \wedge \beta)$ etc.
$\Box$

## (b) Skolem Normal Form

**Recall:** In the proof of CT, we introduced witnessing new constants for existential formulas such that

$\exists x \phi(x)$ is satisfiable iff $\phi(c)$ is satisfiable.

This way an $\exists x$ in front of a formula could be removed at the expense of a new constant.

Now we remove existential quantifiers 'inside' a formula at the expense of extra function symbols:

**16.2 Observation:**
*Let $\phi = \phi(x, y)$ be an $\mathcal{L}$-formula with $x, y \in$Free$(\phi)$. Let $f$ be a new unary function symbol (not in $\mathcal{L}$).*

*Then $\forall x \exists y \phi(x, y)$ is satisfiable iff $\forall x \phi(x, f(x))$ is satisfiable.*
*($f$ is called a* **Skolem function** *for $\phi$.)*

*Proof:* '$\Leftarrow$': clear

'$\Rightarrow$': Let $\mathcal{A}$ be an $\mathcal{L}$-structure with $\mathcal{A} \models \forall x \exists y \phi(x, y)$

$\Rightarrow$ for every $a \in A$ there is some $b \in A$ with $\phi(a, b)$

Interpret $f$ by a function assigning to each $a \in A$ one such $b$
(this uses the Axiom of Choice!). $\square$

**16.3. Example:** $\mathbb{R} \models \forall x \exists y (x \doteq y^2 \vee x \doteq -y^2)$.
Here $f(x) = \sqrt{|\, x \,|}$ will do.

**16.4. Theorem**
*For every $\mathcal{L}$-formula $\phi$*
*there is a formula $\phi^\star$*
*(with new constant and function symbols)*
*having only universal quantifiers in its PNF*
*such that*

$$\phi \text{ is satisfiable iff } \phi^\star \text{ is.}$$

*More precisely,*
*any $\mathcal{L}$-structure $\mathcal{A}$*
*can be made into a structure $\mathcal{A}^\star$*
*interpreting the new constant and function symbols*
*such that*

$$\mathcal{A} \models \phi \text{ iff } \mathcal{A}^\star \models \phi^\star.$$