B1.1 Logic

MT 20

Problem Sheet 3

(1) Show that L_0 and the sequent calculus SQ are equivalent. I.e., using the language $\mathcal{L}_0 = \mathcal{L}[\{\neg, \rightarrow\}]$ of propositional calculus, for all $\Gamma \subseteq \mathrm{Form}(\mathcal{L}_0)$ and for all $\phi \in \mathrm{Form}(\mathcal{L}_0)$:

$$\Gamma \vdash_{L_0} \phi \text{ iff } \Gamma \vdash_{SQ} \phi.$$

(2) The Four Colour Theorem asserts that if a region in the plane is divided into finitely many countries, then each country may be coloured either red, green, blue, or yellow in such a way that no two countries with a common border (of positive length) get the same colour. Use the Compactness Theorem to show that this remains true even if there are countably infinitely many countries.

(3) State and prove the unique readability theorem for predicate calculus
(a) for terms
(b) for atomic formulas
(c) for all formulas.

(4) Let $\mathcal{L} = \{f\}$ be a first-order language containing a unary function symbol f, and no other non-logical symbols. Write down sentences ϕ and ψ of $\mathcal{L}$ such that for any $\mathcal{L}$-structure $\mathcal{A} = \langle A, f_{\mathcal{A}} \rangle$

(i) $\mathcal{A} \models \phi$ if and only if $f_{\mathcal{A}}$ is one-one;

(ii) $\mathcal{A} \models \psi$ if and only if $f_{\mathcal{A}}$ is onto.

(iii) Write down a sentence χ of $\mathcal{L}$ which is satisfiable in some structure with an infinite domain but is false in every structure with a finite domain. What can you say about the size of the domains of the models of the sentence $\neg\chi$?

(iv) Write down a sentence ρ such that whenever $\mathcal{A} \models \rho$ and A is finite, then A contains an even number of elements and, further, every finite set with an even number of elements is the domain of some model of ρ. What can you say about the size of the domains of the models of the sentence $\neg\rho$?

p.t.o.

(5) Let $\mathcal{L} = \{P\}$ be a first-order language with a binary relation symbol P as only non-logical symbol. By exhibiting three suitable $\mathcal{L}$-structures prove (informally) that no two of the following sentences logically implies the other one:

(i) $\forall x \forall y \forall z (P(x, y) \rightarrow (P(y, z) \rightarrow P(x, z)))$,

(ii) $\forall x \forall y (P(x, y) \rightarrow (P(y, x) \rightarrow x \doteq y))$,

(iii) $(\forall x \exists y P(x, y) \rightarrow \exists y \forall x P(x, y))$.

(6) Let $\mathcal{L} = \{f, c\}$ be a first-order language containing as non-logical symbols the unary function symbol f and the constant symbol c. Let $\mathcal{L}_1 = \mathcal{L} \cup \{P\}$, where P is a unary predicate symbol. Consider the following strings (of $\mathcal{L}_1$):

$$\phi : \quad ((P(x_0) \wedge \forall x_0 (P(x_0) \rightarrow P(f(x_2)))) \rightarrow \forall x_2 P(x_0))$$
$$\psi : \quad ((P(c) \wedge \forall x_0 (P(x_0) \rightarrow P(f(x_0)))) \rightarrow \forall x_0 P(x_0))$$

(a) Prove that both ϕ and ψ are formulas of $\mathcal{L}_1$ and indicate the free and bound occurrences of variables in them. Which of these formulas are sentences?

(b) Describe the collection of closed terms of $\mathcal{L}$ and of $\mathcal{L}_1$. (A term is called *closed* if it contains no occurrences of variables.)

(c) Characterise those $\mathcal{L}$-structures $\mathcal{A} = \langle A; f_{\mathcal{A}}; c_{\mathcal{A}} \rangle$ where the domain A is an infinite set, and where $\langle A; f_{\mathcal{A}}; c_{\mathcal{A}}; P_{\mathcal{A}} \rangle \models \psi$ for every unary relation $P_{\mathcal{A}}$ on A.