**B1.1 Logic**

**MT 20**

# Problem Sheet 4

**(1)(i)** Let $\mathcal{L}$ be a first-order language, let $\mathcal{A}$ be an $\mathcal{L}$-structure, let $v$ be an assignment in $\mathcal{A}$ and let $u$ and $t$ be terms in $\mathcal{L}$. Define a new assignment $v'$ by

$$v'(x_j) := \begin{cases} v(x_j) & \text{if } j \neq i \\ \widetilde{v}(t) & \text{if } j = i \end{cases}$$

Let $u[t/x_i]$ be the term obtained by replacing each occurrence of $x_i$ in $u$ by $t$. Show that then $\widetilde{v'}(u) = \widetilde{v}(u[t/x_i])$.

**(ii)** Prove that for any closed (i.e. variable free) terms $t_1, t_2, t_3$ one has $\vdash (t_1 \doteq t_2 \to t_2 \doteq t_1)$ and $\{t_1 \doteq t_2, t_2 \doteq t_3\} \vdash t_1 \doteq t_3$.

**(2)(i)** Prove that $\vdash (\forall x_i(A \to B) \to (\exists x_i A \to B))$ for any formulas $A, B$ provided that the variable $x_i$ does not occur free in $B$.

**(ii)** Let $\phi$ be a formula with just one variable, $x_i$ say, occurring free and let $\Delta$ be a set of sentences. Assume that the constant symbol $c_j$ does not occur in $\phi$ nor in any sentence in $\Delta$, and that $\Delta \vdash \phi[c_j/x_i]$. Sketch a proof that $\Delta \vdash \phi$.

[Hint: First reduce to the case that $\Delta$ is finite and choose $m$ so large that the variable $x_m$ does not occur in $\Delta$ nor in any formula in a derivation of $\phi[c_j/x_i]$ from hyptheses $\Delta$. Then change every occurrence of $c_j$ to $x_m$.]

**(3)** Derive the following theorems:

**(i)** If $x_i$ does not occur free in $A$ then
(a) $\vdash (\exists x_i(A \to B) \to (A \to \exists x_i B))$,
(b) $\vdash ((A \to \exists x_i B) \to \exists x_i(A \to B))$.

**(ii)** If the only variables occurring free in $\phi$ are $x_i$ and $x_j$ (where $i \neq j$) then
(a) $\vdash (\forall x_i \neg\phi \to \neg\forall x_j \phi)$
(b) $\vdash (\exists x_i \forall x_j \phi \to \forall x_j \exists x_i \phi)$.

**p.t.o.**

**(4)** Let $f, g$ be binary function symbols, $P$ a binary predicate symbol, $c, d$ constant symbols and let $\mathcal{L} := \{f, g; P; c, d\}$.

Consider $\mathcal{R} := \langle \mathbb{R}; +, \cdot; <; 0, 1\rangle$ as $\mathcal{L}$-structure. Let $h$ be a unary function symbol, let $\mathcal{L}' := \mathcal{L} \cup \{h\}$ and let $\mathcal{R}'$ be $\mathcal{R}$ together with an interpretation $h_{\mathcal{R}'}$ of $h$ in $\mathcal{R}$.

Find $\mathcal{L}'$-formulas $\phi$ and $\psi$ such that
(i) $\mathcal{R}' \models \phi$ iff $h_{\mathcal{R}'}$ is continuous.
(ii) $\mathcal{R}' \models \psi$ iff $h_{\mathcal{R}'}$ is differentiable.

**(5)(i)** Let $\mathcal{L} := \{+, \cdot; 0, 1\}$ be the language of rings (i.e. $+$ a binary function symbol etc.). Write down sets of formulas $\Phi_p$ (for $p$ a prime or $p = 0$) whose models are exactly all fields of characteristic $p$.

**(ii)** State the Compactness Theorem for sets of sentences and show how it follows from the Soundness and Completeness Theorems.

**(iii)** Prove that $\Phi_0$ in (i) cannot be chosen finite.

**(6)(i)** Axiomatize the first-order theory $\Sigma$ of ordered fields in the language $\mathcal{L} := \{+, \cdot; <; 0, 1\}$.

**(ii)** Which of the following is a model of $\Sigma$:
($\alpha$) $\mathbb{Q}$ with the usual interpretations
($\beta$) $\mathbb{R}$ with the usual interpretations
($\gamma$) $\mathbb{C}$ with $a + bi < c + di$ iff $a^2 + b^2 < c^2 + d^2$
($\delta$) $\mathbb{F}_p$, the field with $p$ (prime) elements with $0 < 1 < 2 < \cdots < p - 1$.

**(iii)** Is $\Sigma$ consistent? Is it maximal consistent?

**(iv)** Recall that the ordering on $\mathbb{Q}$ resp. on $\mathbb{R}$ is *archimedean*, i.e. for every $x \in \mathbb{Q}$ resp. $x \in \mathbb{R}$ there is some $n \in \mathbb{N}$ with $-n < x < n$. Use the Compactness Theorem to prove that archimedeanity is not a first-order property. (Hint: introduce a new constant symbol $c$.)