B8.3 Mathematical Models for Financial Derivatives

Hilary Term 2021

Problem Sheet One

1. Compute the payoffs and sketch the payoff diagrams for the following portfolios involving European options with expiry date T.

   (a) Long one call and two puts, all with strike $K > 0$ (this is a *strip*).

   (b) Long one put and two calls, all with strike $K > 0$ (this is a *strap*).

   (c) Long one put with strike $K_1 > 0$ and long one call with strike $K_2 > K_1$ (this is a *strangle*).

   (d) Long one call with strike $K_1 > 0$, long one call with strike $K_2 > K_1$ and short two calls with strike $K = (K_1 + K_2)/2$ (this is a *butterfly spread*).

   If $S_0 = K$, where $K = (K_1 + K_2)/2$ in (c) (and (d)), what market outcome at T would a speculator be hoping for in each of these portfolios?

2. In 1986 Standard Oil issued bonds for which the bond holder received no interest but at maturity received the face value of the bond, \$1,000 (this is the amount lent by the bond holder to Standard Oil) plus an additional amount, A, which depended on the price of oil at the bond's maturity. This additional amount was set to be 170 times the excess, if any, of the price of a barrel of oil over \$25 subject to the condition that the additional amount, A, could not exceed \$2550.

   (a) Find a formula for A in terms of S_T, the price of a barrel of oil (in dollars per barrel) at the bond's maturity date T.

   (b) Express the additional amount in terms of a long position in a number of call options with strike K_1 and a short position in a number of call options with strike K_2. (Specify the both the number and strikes of these call options, which are on the price of a barrel of oil.)

3. Let $c(S_t, t; K, T)$ denote the price of a European call option with strike $K > 0$, expiry date T, current time $t < T$ and share price is $S_t > 0$ (assume that $S_t > 0$ implies $S_T > 0$). Use no arbitrage arguments to prove the following properties:

   (a) $c(S_t, t; K, T) \leq S_t$,

   (b) $c(S_t, t; K, T) \geq \max(S_t - K\, e^{-r(T-t)}, 0)$, where r is the risk-free rate,

(c) if $0 < K_1 < K_2$ then

$$0 \leq c(S_t, t; K_1, T) - c(S_t, t; K_2, T) \leq (K_2 - K_1)\, e^{-r(T-t)},$$

(d) if $T_1 < T_2$ and $r > 0$, then $c(S_t, t; K, T_1) \leq c(S_t, t; K, T_2)$.

(e) If $0 < K_1 < K_2$ then

$$c(S_t, t; K_1, T) + c(S_t, t; K_2, T) \geq 2C\Big(S_t, t; \frac{K_1 + K_2}{2}, T\Big).$$

4. Consider a two-step binomial model in which at each step the share price either doubles, with probability $p \in (0, 1)$, or halves, with probability $1 - p \in (0, 1)$. Initially the price is $S_0 = 4$. Assume each step takes one unit of time and that over one unit of time the risk-free rate is $r = \log(5/4)$. The possible prices are shown in Figure 1 below.

   (a) Show that the risk-neutral probability for an up-move is $q = \frac{1}{2}$.

   (b) Suppose now, and for the rest of the question, that the option has a payoff which depends on the maximum share price over the life of the option, given by

   $$Y = \max_{t=0,1,2} (S_t - 6)^+.$$

   (This means it is a fixed-strike lookback call.) Compute the final option values $V_2^\omega = Y^\omega$ for each of the outcomes, i.e., possible paths, $\omega \in \Omega = \{uu, ud, du, dd\}$.

   (c) Compute the values of V_1^u and V_1^d, and hence find V_0.

   (d) Find the replicating portfolios $(\phi_1^\omega, \psi_1^\omega)$ for $\omega \in \{u, d\}$.

   (e) Find the replicating portfolio (ϕ_0, ψ_0).

5. (a) Consider a two-step binomial model for a share price with the same properties as in Question 4. An American call is written on this share, with strike $K = 4$. Compute the prices of this American call on the tree and show that they equal the corresponding prices of a European call with strike $K = 4$.

   (b) With the same two-step binomial model as in the previous questions, consider the following American option written on the stock. If the option is exercised at time $t \in 0, 1, 2$ it pays out

   $$Y_t = \left(\frac{1}{1+t}\Big(\sum_{k=0}^{t} S_k\Big) - 2\right)^+,$$

   i.e., it is an American call with fixed strike $K = 2$ on the *average* share price at time t. Find the value of the option on all paths through the tree and determine the optimal exercise strategy (i.e., find the points on paths in the tree where it is optimal to exercise the option).

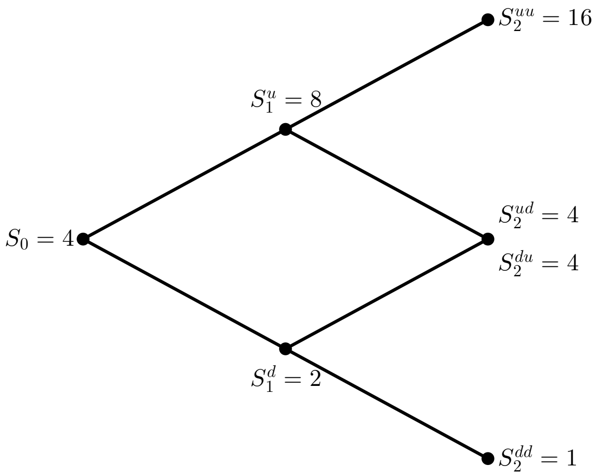

Figure 1: The share-price tree for Questions 4 and 5.

6. Calculate for the price of an at-the-money American Call and an American Put option, in a binomial tree model with 20 steps, where the one-step interest rate is $r = 0.05$ and at each step the price either increases by a factor of 1.1 or decreases by a factor of 0.95. The initial stock price is $S_0 = 100$. (You should not do this calculation by hand).

7. In some markets, for example Oil, there is very little trading in the spot market (i.e. for the underlying commodity), but the forward (or futures) market is highly traded.

   (a) Show that the forward price should be a martingale under the risk-neutral measure.

   (b) Is it possible to replicate an option by trading in the forward market (and bonds)? (justify your answer)

   (c) What would be the advantages and disadvantages to trading in the forward market for Oil, rather than the spot market?

8. Consider a two-step market where the one-step risk-free interest rate r depends on the stock price. The values of the interest rate and the stock are given in Figure 8 below. Calculate the no-arbitrage value of a bond (i.e. a contract with terminal payoff \$1) in each node of the tree.

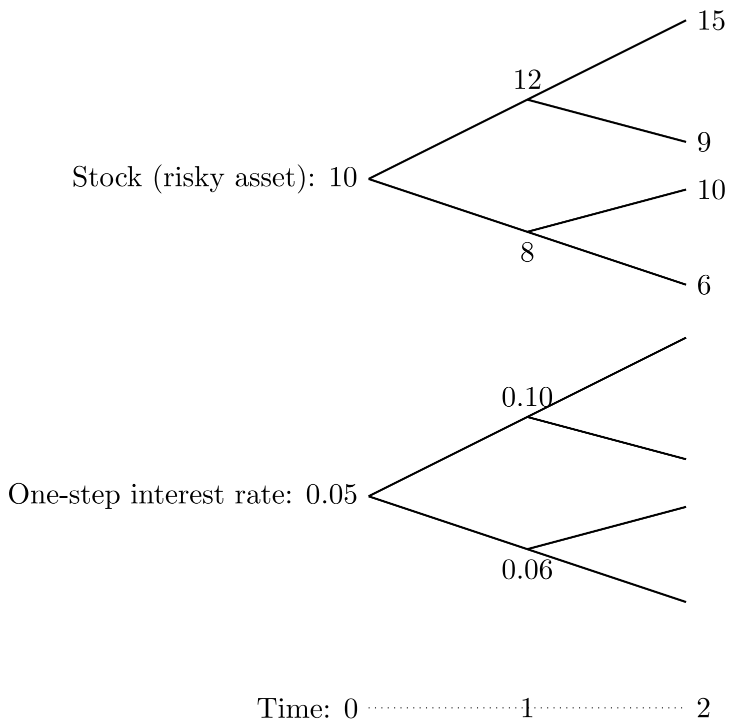

Figure 2: A model with random interest rates