B8.3 Mathematical Models for Financial Derivatives

Hilary Term 2020

Problem Sheet Two

In the following $(W_t)_{t\geq 0}$ denotes a standard Brownian motion and $t \geq 0$ denotes time. A partition π of the interval $[0,t]$ is a sequence of points $0 = t_0 < t_1 < t_2 < \cdots < t_n = t$ and $|\pi| = \max_k(t_{k+1} - t_k)$. On a given partition $W_k \equiv W_{t_k}$, $\delta W_k \equiv W_{k+1} - W_k$, $\delta t_k \equiv t_{k+1} - t_k$ and if f is a function on $[0,t]$, $f_k \equiv f(t_k)$ and $\delta f_k \equiv f_{k+1} - f_k$.

1. Show that if $t, s \geq 0$ then $\mathbb{E}[W_s\, W_t] = \min(s,t)$.

2. Assuming that both the integral and its variance exist, show that

$$\operatorname{var}\Big[\int_0^t f(W_s,s)\,dW_s\Big] = \int_0^t \mathbb{E}\big[f(W_s,s)^2\big]\,ds.$$

   Is it generally the case that $\int_0^t f(W_s,s)dW_s$ has a Gaussian distribution?

   [Note: if the integral and its variance exist then it is legitimate to interchange the order of expectation and dt-integration and the stochastic integral is a martingale.]

3. Use the differential version of Itô's lemma to show that

   (a) $\displaystyle\int_0^t W_s\,ds = t\,W_t - \int_0^t s\,dW_s = \int_0^t (t-s)\,dW_s,$

   (b) $\displaystyle\int_0^t W_s^2\,dW_s = \tfrac{1}{3}W_t^3 - \int_0^t W_s\,ds,$

4. Define X_t to be the 'area under a Brownian motion', $X_0 = 0$ and $X_t = \int_0^t W_u\,du$ for $t > 0$. Show that X_t is normally distributed with

$$\mathbb{E}\big[X_t\big] = 0, \quad \mathbb{E}\big[X_t^2\big] = \tfrac{1}{3}\,t^3.$$

   Now define Y_t as the 'average area under a Brownian motion',

$$Y_t = \begin{cases} 0 & \text{if } t = 0, \\ X_t/t & \text{if } t > 0. \end{cases}$$

   Show that Y_t has $\mathbb{E}[Y_t] = 0$, $\mathbb{E}\big[Y_t^2\big] = t/3$ and that Y_t is continuous for all $t \geq 0$.

   Is $\sqrt{3}\,Y_t$ a Brownian motion? Give reasons for your answer.

5. Find solutions of the Black–Scholes terminal value problem

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-y)\, S\, \frac{\partial V}{\partial S} - r\, V = 0, \quad S > 0,\ t < T,$$

$$V(S,T) = f(S), \quad S > 0,$$

when

(a) $f(S) = C$, where C is a constant;

(b) $f(S) = S^\alpha$, where α is a constant.

[Hint: you don't need the Feynman–Kac formula to do either of these. Look for simple functional forms of the solution.]

6. Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfy the PDE

$$\frac{\partial f}{\partial t} + 2(|x| + t)\frac{\partial f}{\partial x} + \frac{1}{1+\exp(x)}\frac{\partial^2 f}{\partial x^2} = 0$$

(a) If f has boundary values $f(x,T) = g(x)$ for all x, write down an SDE for X such that $f(x,t) = \mathbb{E}[g(X_T)|X_t = x]$.

(b) If f has boundary values $f(x,T) = g(x)$ for all $x > 0$ and $f(0,t) = 0$ for all $t > 0$, and $g(0) = 0$, write down an SDE for Y such that $f(y,t) = \mathbb{E}[g(Y_T)|Y_t = y]$.

7. An Ornstein–Uhlenbeck process X is the solution to the stochastic differential equation

$$dX_t = \kappa(\theta - X_t)dt + \sigma dW_t$$

where $\kappa > 0$ and $\theta \in \mathbb{R}$.

(a) By using Itô's lemma applied to $e^{\kappa t}X_t$, show that for a given initial value X_0, the value of X_t is given by

$$X_t = \theta + \Big((X_0 - \theta) + \sigma \int_0^t e^{\kappa s} dW_s\Big)e^{-\kappa t}$$

(b) Show that this implies that, for any deterministic initial value X_0, X_t has a Gaussian distribution, with mean and variance you should determine.

(c) Calculate $f(x,t) = \mathbb{E}[X_T^2|X_t = x]$, and check explicitly that this is a solution to the corresponding PDE:

$$\frac{\partial f}{\partial t} + \kappa(\theta - x)\frac{\partial f}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2} = 0.$$

Optional questions

8. The Black–Scholes equation from a binomial method.

   One step of the Cox, Ross & Rubinstein binomial method can be written as
$$V(S,t) = e^{-r\delta t}\Big(q\,V^u + (1-q)\,V^d\Big)$$
   where
$$V^u = V\big(S^u, t+\delta t\big), \quad V^d = V\big(S^d, t+\delta t\big),$$
$$S^u = S\,e^{\sigma\sqrt{\delta t}}, \quad S^d = S\,e^{-\sigma\sqrt{\delta t}}, \quad q = \frac{e^{r\delta t} - e^{-\sigma\sqrt{\delta t}}}{e^{\sigma\sqrt{\delta t}} - e^{-\sigma\sqrt{\delta t}}},$$
   $\sigma > 0$ is the volatility, r is the risk-free rate and $\delta t > 0$ is the length of the time-step. Supposing this is true for all $S > 0$ and that $V(S,t)$ may be expanded in a Taylor series in both S and t, show that as $\delta t \to 0$
$$\begin{aligned} q &= \frac{1}{2} + \frac{r - \frac{1}{2}\sigma^2}{2\,\sigma}\sqrt{\delta t} + \mathcal{O}(\delta t), \\ V^u &= V + \sqrt{\delta t}\,\sigma\,S\,\frac{\partial V}{\partial S} + \delta t\left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2\Big(S\,\frac{\partial V}{\partial S} + S^2\,\frac{\partial^2 V}{\partial S^2}\Big)\right) + \mathcal{O}(\delta t^{3/2}), \\ V^d &= V - \sqrt{\delta t}\,\sigma\,S\,\frac{\partial V}{\partial S} + \delta t\left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2\Big(S\,\frac{\partial V}{\partial S} + S^2\,\frac{\partial^2 V}{\partial S^2}\Big)\right) + \mathcal{O}(\delta t^{3/2}), \end{aligned}$$
   where V and all its partial derivatives are evaluated at (S,t). Hence show that in the limit $\delta t \to 0$ the option price satisfies the (zero dividend-yield) Black–Scholes equation
$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2\,S^2\,\frac{\partial^2 V}{\partial S^2} + r\,S\,\frac{\partial S}{\partial S} - r\,V = 0.$$

9. The total variation of a function, or stochastic process, over $[0,t]$, is
$$\langle f\rangle_t = \lim_{|\pi|\to 0}\sum_{k=0}^{n-1} |\,f_{k+1} - f_k\,|.$$
   If $\langle f\rangle_t$ is finite on $[0,t]$ we say f has bounded variation on $[0,t]$. Show that:

   (a) if f is $C^1[0,t]$ then $\langle f\rangle_t = \int_0^t |f'(t)|\,dt < \infty$;

   (b) if f is a continuous function with $\langle f\rangle_t < \infty$ then its quadratic variation is zero, $[\,f\,]_t = 0$;

   (c) Brownian motion does not have bounded variation;

(d) the arc length of the graph of a Brownian motion is infinite for any $t > 0$.

[Hint: if $y = X_t$ has an arc length s then $ds = \sqrt{dy^2 + dx^2} \geq \sqrt{dy^2} = |dy|$.]

10. Let X be the solution to an SDE of the form

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t.$$

If S is a function satisfying the ODE

$$b(x)S'(x) + \frac{1}{2}(\sigma(x))^2 S''(x) = 0$$

find the dynamics of $Y_t = S(X_t)$ and an expression for its quadratic variation $[Y]_t$. What do you conclude about the behaviour of Y?

11. The covariation of two functions or processes, X and Y, on $[0,t]$ is defined to be

$$[X,Y]_t = \lim_{|\pi|\to 0} \sum_{k=0}^{n-1} (X_{k+1} - X_k)(Y_{k+1} - Y_k).$$

Show that if both X and Y have finite quadratic variation on $[0,t]$ then $[X,Y]_t$ is finite and satisfies $2\big|\,[X,Y]_t\,\big| \leq [X]_t + [Y]_t$.

Assuming $[X]_t$ and $[Y]_t$ are finite, show that

(a) $[X+Y]_t = [X]_t + [Y]_t + 2\,[X,Y]_t$,

(b) $[X,Y]_t = \frac{1}{4}\big([X+Y]_t - [X-Y]_t\big)$.

(c) if X and Y are C^1 functions on $[0,t]$ then $[X,Y]_t = 0$.

12. Let $(W_t)_{t\geq 0}$ and $(Z_t)_{t\geq 0}$ be two Brownian motions. They are correlated with correlation $\rho \in (-1,1)$ if

(a) for all $s,t \geq 0$, $\mathbb{E}\big[(W_{t+s} - W_t)(Z_{t+s} - Z_t)\big] = \rho\, s$,

(b) for all $0 \leq p \leq q \leq s \leq t$, the pair $(W_q - W_p)$ and $(Z_t - Z_s)$ are independent and the pair $(W_t - W_s)$ and $(Z_q - Z_p)$ are also independent.

Show that in this case $[W,Z]_t = \rho\, t$, in the sense that

$$\mathbb{E}\big[[W,Z]_t - \rho\, t\big] = 0 \quad \text{and} \quad \mathbb{E}\Big[\big([W,Z]_t - \rho\, t\big)^2\Big] = 0.$$

[Hint: first show that if X and Y are random variables with second moments then $|\,\mathbb{E}[\,XY\,]\,| \leq \frac{1}{2}\big(\mathbb{E}[X^2] + \mathbb{E}[Y^2]\big)$.]

[Note that if we define a process by $f_t = f(W_t, Z_t, t)$ where $f(W, Z, t)$ is $C^{2,2,1}$, then (the differential version of) Itô's lemma is

$$\begin{aligned} df_t &= \frac{\partial f}{\partial t}\,dt + \frac{\partial f}{\partial W}\,dW_t + \frac{\partial f}{\partial Z}\,dZ_t \\ &+ \tfrac{1}{2}\frac{\partial^2 f}{\partial W^2}\,d[W]_t + \tfrac{1}{2}\frac{\partial^2 f}{\partial Z^2}\,d[Z]_t + \frac{\partial^2 f}{\partial W \partial Z}\,d[W, Z]_t, \end{aligned}$$

where all functions on the right-hand side are evaluated at (W_t, Z_t, t). The result derived above simplifies this expression.]