B8.3 Mathematical Models for Financial Derivatives

Hilary Term 2021

Problem Sheet Three

1. A log-option is an option with the payoff function

$$P_{\mathrm{o}}(S_T) = \log(S_T/K),$$

where the "strike" is positive, $K > 0$. Find the Black–Scholes value function for a European log-option. (Such options are not traded, but they occur in the theory underlying the CBOE's VIX (variance index) which measures the S&P500 index's variance, allowing futures and options to be written on this variance.)

2. (a) Find the Black–Scholes price function of a European digital call option, i.e., an option whose payoff function is

$$f(S_T) = \mathbf{1}_{\{S_T \geq K\}} = \begin{cases} 0 & \text{if } 0 < S_T < K, \\ 1 & \text{if } S_T \geq K. \end{cases}$$

   (b) A European digital put option has the payoff $f(S_T) = \mathbf{1}_{\{S_T < K\}}$. Use a no arbitrage argument to establish a digital put-call parity result and hence find the Black–Scholes price function for a digital put.

   (c) Calculate the $\Delta$ and $\Gamma$ of the digital call option. Describe the behaviour of the $\Delta$ as $t \to T$, particularly when $S \approx K$, and consider the implications of this on hedging these options in practice.

3. Show that if $V(S,t)$ is a sufficiently differentiable solution of the Black–Scholes equation (for $S > 0$ and $t < T$) then so too is

$$W(S,t) = S\,\frac{\partial V}{\partial S}(S,t).$$

By induction, conclude that if $V(S,t)$ is sufficiently differentiable then

$$\left(S\,\frac{\partial}{\partial S}\right)^n V(S,t) \quad \text{and} \quad S^n\,\frac{\partial^n V}{\partial S^n}(S,t), \qquad n = 2,3,4,\ldots$$

are also solutions of the Black–Scholes equation.

4. Show that if $V(S,t)$ is a solution of the Black–Scholes equation (for $S > 0$ and $t < T$) and $B > 0$ then

$$W(S,t) = \left(\frac{S}{B}\right)^{2\alpha} V\left(\frac{B^2}{S}, t\right),$$

where $2\alpha = 1 - 2(r-y)/\sigma^2$, is also a solution of the (same) Black–Scholes equation.

[Hint: put $\xi = B^2/S$ and note that $V(\xi, t)$ satisfies the Black–Scholes equation in $\xi > 0$ and $t < T$.]

5. Suppose we have a Black–Scholes model with $r = q = 0$ and $\sigma = 0.4$, and an initial stock price of $S_0 = 100$. A trader sets up a $\Delta$-hedged portfolio consisting of one (long) at the money call option (with expiry $T = 1$), and an appropriate number of stocks. They then do not change this portfolio until time $T = 1$.

   (a) Calculate the number of stocks which should be held in the portfolio, and the initial amount of cash held.

   (b) Describe the payoff of this portfolio at time $T$, and give the scenarios (in terms of the value of $S_T$) in which this strategy will turn a profit.

6. Use the CBOE webpage

   `https://www.cboe.com/delayed_quotes/goog/quote_table`

   to obtain current quotes for Call options on Google (GOOG) expiring in January 2022, excluding all options with a log-moneyness below $-0.2$ or above $0.2$. Compute the mid price (average of bid and ask prices) for these options. Using a risk-free rate of $r = 0.01$ and assuming no dividends, compute the implied volatility for each option (do not use the CBOE provided IVs), and plot this against the negative log-moneyness of the option. Do you observe a smile? A skew?

   (If working in excel, the 'solver' add-in may be useful to compute implied volatilities, if the solver has difficulty computing the implied volatilites for options far out of the money, then exclude them. You may find it easier to compute using Matlab, R or Python, if you are comfortable with one of these languages.)

## Optional questions

7. Show that if $V(S, t)$ is a solution of the Black–Scholes equation (for $S > 0$ and $t < T$) then so too are:

   (a) $a\,V(S, t)$ with $a \in \mathbb{R}$;

   (b) $V(bS, t)$ with $b > 0$;

   (c) $a\,V(bS, t)$ with $a \in \mathbb{R}$, $b > 0$.

8. Let $C_{\mathrm{bs}}(S, t; K, T, r, y, \sigma)$ denote the solution of a Black–Scholes call value problem with strike $K$, expiry date $T$, risk-free rate $r$, continuous dividend yield $q$ and volatility $\sigma$. Consider the Black–Scholes problem

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q)\, S \frac{\partial V}{\partial S} - r\, V = 0, \quad S > 0,\ t < T$$

$$V(S, t) = \frac{1}{K^2} \max\bigl(S^3 - K^3, 0\bigr), \quad S > 0.$$

Show that

$$V(S, t) = \frac{1}{K^2} C_{\mathrm{bs}}(S^3, t; K^3, T, r, \hat{y}, \hat{\sigma})$$

where $\hat{q} = 3q - 2r - 3\sigma^2$ and $\hat{\sigma} = 3\sigma$.

[Hint: either write $\hat{S} = S^3$ and do a change of variables in the terminal value problem or think about the payoff and risk-neutral dynamics for $\hat{S}_t = S_t^3$.]