B8.3 Mathematical Models for Financial Derivatives

Hilary Term 2021

Problem Sheet Four

1. Suppose that $V(S,t)$ satisfies the Black–Scholes problem

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)\, S \frac{\partial V}{\partial S} - r\, V = 0, \quad S > 0,\ t < T,$$
$$V(S,T) = P_{\text{o}}(S), \quad S > 0.$$

Use the chain rule to show that if $F = S\, e^{(r-q)(T-t)}$ (the forward price of S over the time interval $[t,T]$), $t' = t$ and $\hat{V}(F,t') = V(S,t)$ then

$$\frac{\partial \hat{V}}{\partial t'} + \tfrac{1}{2}\sigma^2 F^2 \frac{\partial^2 \hat{V}}{\partial F^2} - r\, \hat{V} = 0, \quad F > 0,\ t' < T,$$
$$\hat{V}(F,T) = P_{\text{o}}(F), \quad F > 0.$$

2. Suppose that the price of a share evolves according to

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t, \quad S_0 > 0,$$

and that the share does *not* pay any dividends. There is a constant, risk-free, continuously-compounded interest rate r.

A non-standard European derivative security is written on this share; in addition to paying the up-front price of the derivative the holder of the claim must also pay an amount $\beta\, S_t^{\alpha}\, dt$ over each interval $[t, t+dt)$ during the life of the derivative, for some $\alpha \in \mathbb{R}$.

Let the derivative security's price function be $V(S,t)$, for $S > 0$ and $t \le T$.

(a) By suitably adapting either the hedging or the self-financing replication argument, show that $V(S,t)$ must satisfy the partial differential equation

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r\, S \frac{\partial V}{\partial S} - r\, V = \beta\, S^{\alpha}.$$

(b) Assume first that $\beta = 0$. Find all separable solutions of the form $V(S,t) = f(t)\, S^{\gamma}$ where γ is a real constant. Assume that $f(T) = 1$.

(c) Find the steady-state solutions of the PDE, that is, find solutions $V(S)$ that only depend on S. Assume here that $\alpha \ne 1$, $\alpha \ne -2r/\sigma^2$ and $\beta \ne 0$.

(d) Without doing the details, briefly explain how you would solve the problem

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r\, S\, \frac{\partial V}{\partial S} - r\, V = \beta\, S^{\alpha},$$
$$V(S,T) = S^3,$$

assuming again that $\alpha \neq 1$, $\alpha \neq -2r/\sigma$ and $\beta \neq 0$.

3. Consider the following perpetual American option problem. The option's payoff is

$$P_o(S) = \begin{cases} K - S/3 & \text{if } 0 < S \leq K, \\ 0 & \text{if } S > K. \end{cases}$$

Assume that the option value satisfies the steady-state Black–Scholes equation

$$\mathcal{L}_{\text{ssbs}}[V] = \tfrac{1}{2}\sigma^2 S^2\, V''(S) + (r-q)\, S\, V'(S) - r\, V = 0, \quad \hat{S} < S,$$

where $0 < \hat{S} \leq K$ is the optimal exercise boundary and where $\sigma > 0$, $r > 0$ and $q > 0$ are constants. The option satisfies the boundary conditions

$$V(\hat{S}) = K - \hat{S}/3, \qquad \lim_{S\to\infty} V(S) \to 0.$$

(a) Give a *sketch* of the payoff and option price as functions of S and indicate where $\mathcal{L}_{\text{ssbs}}[V] = 0$, where $\mathcal{L}_{\text{ssbs}}[V] < 0$, where $V(S) > P_o(S)$ and where $V(S) = P_o(S)$.

(b) Show that, under the assumptions given above, the quadratic

$$p(m) = \tfrac{1}{2}\sigma^2\, m(m-1) + (r-q)\, m - r$$

has two distinct real roots and only one of these is strictly negative.

(c) Assume that we have smooth pasting at $\hat{S}$, i.e., $V'(\hat{S}) = -1/3$. Show that this implies that

$$\hat{S} = \frac{3m^-}{m^- - 1}\, K,$$

where $m^- < 0$ is the negative root of the quadratic $p(m)$.

(d) Show that smooth pasting only makes sense if $-\frac{1}{2} < m^- < 0$.

(e) What is the optimal exercise boundary if $m^- < -\frac{1}{2}$? Justify your answer.

(f) Suppose that $-\frac{1}{2} < m^- < 0$, so that smooth pasting does give the correct optimal exercise boundary. Suppose also that the holder of the option decides that they are going to ignore the optimal exercise boundary $\hat{S}$ and simply exercise the option as soon as $S \leq \bar{S}$ where $0 < \bar{S} < K$ is chosen by the holder. In this case the value of the option, $\bar{V}(S,t)$, satisfies the problem

$$\mathcal{L}_{\text{ssbs}}[\bar{V}] = 0, \quad S > \bar{S},$$
$$\bar{V}(\bar{S}) = K - \bar{S}/3, \quad \lim_{S\to\infty} \bar{V}(S) \to 0.$$

Find $\bar{V}(S)$ and show that

i. if $\bar{S} > \hat{S}$ then one could increase the value of the option by decreasing $\bar{S}$ (hint; differentiate with respect to $\bar{S}$);

ii. if $\bar{S} < \hat{S}$ then there is a potential arbitrage in the price $\bar{V}(S)$ (hint; differentiate with respect to S).

4. Let T_1 and T_2 be given times with $0 < T_1 < T_2$ and let $\alpha > 0$ be a given constant. A forward-start put is a European put option written on an asset whose price is S_t, but where the strike is not given at time zero, rather it is set equal to $\alpha\, S_{T_1}$, where S_{T_1} is the share price at time T_1. Find the option price and Δ for $T_1 < t < T_2$ and then for $0 \leq t \leq T_1$.

5. An up-and-out barrier put option is an option which has the payoff of a regular put option provided the share price stays below a barrier, $B > 0$, for the life of the option, i.e., provided $S_t < B$ for all $t \in [0, T]$. If at any time $t \in [0, T]$ we have $S_t \geq B$ the option immediately becomes worthless.

   (a) Write down the Black–Scholes problem for the price function of this option assuming that the share price has stayed below the barrier.

   (b) Find the Black–Scholes value function for this option in terms of a vanilla put's value function assuming that the barrier lies above the strike, $0 < K < B$.

   (c) Find the Black–Scholes value function for this option in terms of the price functions for vanilla and digital puts assuming the barrier lies below the strike, $0 < B < K$.

   (d) By analogy with the down-and-in call option, define an up-and-in put and find a formula for its value in the case $0 < K < B$.

6. A down-and-out digital call option is a digital call option which becomes worthless if $S_t \leq B$ at any time during the options life, $[0, T]$. Here $B > 0$ is called the barrier. If we have $S_t > B$ for all $t \in [0, T]$

then the payoff for the option is the unit step function $\mathbf{1}_{\{S_T \geq K\}}$. If the underlying share pays a constant, continuous dividend yield q, find the Black–Scholes value of such an option if:

(a) the barrier is less than the strike, $0 < B < K$;

(b) the barrier is greater than the strike, $0 < K < B$.

7. If an option depends on a continuously sampled arithmetic average of the share price we can write its price as $V_t = V(S_t, R_t, t)$ where $R_t = \int_0^t S_u \, du$ and at expiry the average share price is $A_T = R_T/T$. The Black–Scholes equation for the function $V(S, R, T)$ is

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - y)\, S \frac{\partial V}{\partial S} + S \frac{\partial V}{\partial R} - r\, V = 0.$$

For $t < T$, find the solution of this equation if the payoff is

$$V(S, R, T) = A\, S + B\, R + C$$

for constants A, B and C. Assume that $r \neq q$.

[Hint: try a solution of the form $V(S, R, t) = a(t)\, S + b(t)\, R + c(t)$.]

Assume now that the payoff is $R_T = T \times A_T$. Explain how you could perfectly hedge such a contract. Is your method independent of the Black–Scholes equation?

Optional questions

8. **Time-dependent parameters.**

   Assume that the price of a share, S_t, evolves according to the SDE

   $$\frac{dS_t}{S_t} = \big(\,\bar{\mu}(t) - \bar{q}(t)\,\big)\,dt + \bar{\sigma}(t)\,dW_t,$$

   where $\bar{\mu}(t)$, $\bar{q}(t)$ and $\bar{\sigma}(t) > 0$ are known functions of time. Assume also that the risk-free rate is a known function of time, $\bar{r}(t)$.

   (a) Derive the Black–Scholes problem, for $S > 0$,

   $$\begin{aligned}\frac{\partial C}{\partial t} + \tfrac{1}{2}\bar{\sigma}(t)^2 S^2 \frac{\partial^2 C}{\partial S^2} + \big(\bar{r}(t) - \bar{q}(t)\big) S \frac{\partial C}{\partial S} - \bar{r}(t) C = 0,\ t < T,\\ V(S,T) = (S-K)^+,\end{aligned} \tag{1}$$

   for the value (function) $C(S,t)$ of a European call option written on the share.

   (b) Use the Feynman–Kac theorem to show that

   $$C(S,t) = \exp\Big(-\int_t^T \bar{r}(u)\,du\Big)\,\mathbb{E}_t\big[\,(S_T - K)^+ \mid S_t = S\,\big],$$

   where S_t evolves as

   $$\frac{dS_t}{S_t} = \big(\bar{r}(t) - \bar{q}(t)\big)\,dt + \bar{\sigma}(t)\,dW_t. \tag{2}$$

   (c) Deduce that the solution to (2) is

   $$S_T = S_t\,\exp\left(\int_t^T \big(\bar{r}(u) - \bar{q}(u) - \tfrac{1}{2}\bar{\sigma}(u)^2\big)du + \int_t^T \bar{\sigma}(u)\,dW_u\right).$$

   (d) Hence deduce that for fixed $t < T$ the solution of (1) is

   $$C(S,t) = C_{\mathrm{bs}}(S,t;K,T,\hat{r},\hat{q},\hat{\sigma})$$

   where

   $$\hat{r} = \frac{1}{T-t}\int_t^T \bar{r}(u)\,du,\ \hat{q} = \frac{1}{T-t}\int_t^T \bar{q}(u)\,du,\ \hat{\sigma}^2 = \frac{1}{T-t}\int_t^T \bar{\sigma}(u)^2\,du,$$

   and $C_{\mathrm{bs}}(S,t;K,T,r,q,\sigma)$ is the Black–Scholes formula for a call with strike K, expiry T, constant risk-free rate r, constant continuous dividend yield q and constant volatility σ.

9. A European log-put option has the payoff

$$V_T = \big(-\log(S_T/K)\big)^+$$

(a) Show that if S_u evolves as

$$\frac{dS_u}{S_u} = r\,du + \sigma\,dW_u,\ t < u \le T, \quad S_t = S,$$

then

$$\text{prob}(S_T < K) = \text{N}(-d_-), \quad d_- = \frac{\log(S/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sqrt{\sigma^2\,(T-t)}}.$$

(b) Assuming the underlying share pays no dividends, show that the Black–Scholes value function for the log-put is

$$V(S,t) = e^{-r(T-t)}\sqrt{\sigma^2\,(T-t)}\,\Big(d_-\,\text{N}(-d_-) - e^{-\frac{1}{2}d_-^2}/\sqrt{2\pi}\Big).$$

10. Let $0 < T_1 < T_2$ and $K > 0$. A derivative security with the following properties is written on a share (which does not pay any dividends between time $t = 0$ and $t = T_2$). If at time T_1 the share price is greater than or equal to K, $S_{T_1} \ge K$, then the derivative security becomes a European call option with strike S_{T_1} and expiry date T_2. If $S_{T_1} < K$, it becomes a European put option with strike S_{T_1} and expiry date T_2. Find the Black–Scholes price function for this security when $T_1 < t < T_2$ and then when $0 \le t \le T_1$.

11. Assume that the USD/GBP exchange rate, X_t, evolves according to the SDE

$$\frac{dX_t}{X_t} = \mu\,dt + \sigma\,dW_t.$$

(a) Given that today's exchange rate, at $t = 0$, is X_0 find the expected USD/GBP exchange rate $\mathbb{E}[X_T]$ at time $T > 0$.

(b) Find the SDE which the GBP/USD exchange rate, $Y_t = 1/X_t$ follows.

(c) Given that $Y_0 = 1/X_0$ today, find the expected GBP/USD exchange rate $\mathbb{E}[Y_T]$ at time $T > 0$.

(d) Show that, although $X_T\,Y_T = 1$ for any $T > 0$,

$$\mathbb{E}[X_T]\,\mathbb{E}[Y_T] = e^{\sigma^2 T}.$$

12. An investor has the choice of investing their wealth of 1 unit of currency in either a risky asset whose price evolves as

$$\frac{dS_t}{S_t} = \mu\,dt + \sigma\,dW_t,\ t > 0, \quad S_0 = 1,$$

where $\sigma > 0$, or in a risk-free bond whose price evolves as

$$\frac{dB_t}{B_t} = r\,dt, \; t > 0, \quad B_0 = 1,$$

where $0 < r < \mu - \frac{1}{2}\sigma^2$. The investment horizon is $[0, T]$. The investor decides to invest their funds in the risky asset, but is worried that when they withdraw the funds, at time T, the risk-free bonds may have outperformed the risky assets. So they consider the possibility of purchasing a put option with maturity T to protect themselves against this possibility. (They borrow money to buy the put.)

(a) What is the probability of the risky asset underperforming the risk-free one, i.e, what is the probability that $S_T < e^{rT}$?

(b) What happens to this probability as $T \to \infty$?

(c) What should the strike of the put be in order that the investor is completely insured against the possibility of underperformance?

(d) What happens to the price of the insurance as $T \to \infty$?