B8.2 Continuous Martingales and Stochastic Calculus

Filtrations and stopping times

Stopped processes

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It is often useful to be able to 'stop' a process at a stopping time and know that the result still has nice measurability properties.

Theorem

Let X be a progressively measurable process and τ a stopping time. Then $X_\tau 1_{\tau<\infty}$ is $\mathscr{F}_\tau$-measurable. The stopped process $X^\tau = (X_{\tau\wedge t} : t \geq 0)$ is progressively measurable (where $X_\tau 1_{\tau<\infty}(\omega) = X_{\tau(\omega)}(\omega) 1_{\tau(\omega)<\infty}$).

Proof.

The first statement is 'easy' once we prove X^τ is progressive. Observe $X_{\tau\wedge s}$ on $[0,t]\times\Omega$ is a composition of two maps

$$(s,\omega)\mapsto(\tau(\omega)\wedge s,\omega),$$
$$([0,t]\times\Omega,\mathscr{B}([0,t])\otimes\mathscr{F}_t)\mapsto([0,t]\times\Omega,\mathscr{B}([0,t])\otimes\mathscr{F}_t)$$

and

$$(u,\omega)\mapsto X_u(\omega),$$
$$([0,t]\times\Omega,\mathscr{B}([0,t])\otimes\mathscr{F}_t)\mapsto(\mathbb{R},\mathscr{B}(\mathbb{R})),$$

both of which are measurable since τ is a stopping time and X is progressively measurable. $\square$

- In other areas of analysis, we often prove results by showing that a statement holds 'locally'.
- This is made difficult for stochastic processes by the fact that these depend on the random seed ω, for which we have no topology.
- One useful way around this challenge is to use stopping times, which helpfully 'localize' in both time and space.

Definition

We say that a process X *locally* has some property C if there exists a sequence of stopping times $\{\tau_n\}_{n\in\mathbb{N}}$ such that the stopped processes X^{τ_n} have property C for every n, and $\tau_n \uparrow \infty$ almost surely. The sequence τ_n is said to *localize* or *reduce* X.

Example

A Brownian motion is *locally bounded*, but is not bounded overall.

When we work with processes locally, it is then useful to reconstruct the 'global' process from its local versions.

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Lemma

Given a localizing sequence $\{\tau_n\}_{n\in\mathbb{N}}$ and a family of processes $\{Y^n\}_{n\in\mathbb{N}}$ such that, for all $n \leq m$,

$$1_{t\leq\tau_n} Y^n = 1_{t\leq\tau_n} Y^m$$

up to indistinguishability, there exists a process X such that $1_{t\leq\tau_n} Y^n = 1_{t\leq\tau_n} X$ for all n.

Proof.

We can construct X explicitly by

$$X_t = \sum_{n\in\mathbb{N}} 1_{\tau_n < t \leq \tau_{n+1}} Y^{n+1}$$

(many constructions are possible). □