B8.2 Continuous Martingales and Stochastic Calculus

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Filtrations and stopping times

Stopping times

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Again the definition mirrors what you know from the discrete setting.

Definition

Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathscr{P})$ be a filtered space. A random variable $\tau : \Omega \mapsto [0, +\infty]$ is called a *stopping time* (relative to $\{\mathscr{F}_t\}_{t\geq 0}$) if

$$\{\tau \leq t\} \in \mathscr{F}_t \qquad \text{for all } t \geq 0.$$

Stopping times are sometimes called optional times (or example, in the 'optional stopping theorem').

The 'first time a certain phenomenon occurs' will be a stopping time. Our fundamental examples will be first hitting times of sets.

If X is a stochastic process and $\Gamma \in \mathscr{B}(\mathbb{R})$ we set

$$H_\Gamma(\omega) := \inf\{t \geq 0 : X_t(\omega) \in \Gamma\}. \tag{1}$$

Exercise

Show that

1. *if X is adapted to $\{\mathscr{F}_t\}_{t\geq 0}$ and has right-continuous paths then H_Γ, for Γ an open set, is a stopping time relative to $(\mathscr{F}_{t+})$.*
2. *if X has continuous paths, then H_Γ, for Γ a closed set, is a stopping time relative to $\{\mathscr{F}_t\}_{t\geq 0}$.*

One can show that the hitting time of any Borel set, or of a (reasonably nice) set which changes in time, is a stopping time (assuming $\{\mathscr{F}_t\}_{t\geq 0}$ satisfies the usual conditions), but this is surprisingly difficult!

With a stopping time we can associate 'the information known at time τ':

Definition

Given a stopping time τ relative to $\{\mathscr{F}_t\}_{t\geq 0}$ we define

$$\mathscr{F}_\tau := \{A \in \mathscr{F} : A \cap \{\tau \leq t\} \in \mathscr{F}_t \forall t \geq 0\},$$
$$\mathscr{F}_{\tau-} := \sigma(\{A \cap \{\tau > t\} : t \geq 0, A \in \mathscr{F}_t\})$$

(which satisfy all the natural properties).

Proposition

Let τ be a stopping time. Then

(i) $\mathscr{F}_{\tau-}$ *and* $\mathscr{F}_{\tau}$ *are* σ*-algebras and* τ *is* $\mathscr{F}_{\tau-}$*-measurable.*

(ii) $\mathscr{F}_{\tau-} \subseteq \mathscr{F}_{\tau}$

(iii) *If* $\tau = t$ *then* $\mathscr{F}_{\tau} = \mathscr{F}_{t}$

(iv) *If* τ *and* ρ *are stopping times then so are* $\tau \wedge \rho$, $\tau \vee \rho$ *and* $\tau + \rho$ *and* $\{\tau \leq \rho\} \in \mathscr{F}_{\tau \wedge \rho}$. *Further if* $\tau \leq \rho$ *then* $\mathscr{F}_{\tau} \subseteq \mathscr{F}_{\rho}$.

(v) *If* τ *is a stopping time and* ρ *is a* $[0, \infty]$*-valued random variable which is* $\mathscr{F}_{\tau}$*-measurable and* $\rho \geq \tau$, *then* ρ *is a stopping time. In particular,*

$$\tau_n := \sum_{k=0}^{\infty} \frac{k+1}{2^n} 1_{\{\frac{k}{2^n} < \tau \leq \frac{k+1}{2^n}\}} + \infty 1_{\{\tau = \infty\}} \tag{2}$$

is a sequence of stopping times with $\tau_n \downarrow \tau$ *as* $n \to \infty$.

Proof.

We prove (v):

Note that $\{\rho \leq t\} = \{\rho \leq t\} \cap \{\tau \leq t\} \in \mathscr{F}_t$ since ρ is $\mathscr{F}_\tau$-measurable. Hence ρ is a stopping time. We have $\tau_n \downarrow \tau$ by definition, and clearly τ_n is $\mathscr{F}_\tau$-measurable since τ is $\mathscr{F}_\tau$-measurable. $\square$

Lemma

For any integrable random variable X, any stopping times ρ and τ,

$$1_{\rho\leq\tau}\mathbb{E}[X|\mathscr{F}_\rho] = 1_{\rho\leq T}\mathbb{E}[X|\mathscr{F}_{\rho\wedge\tau}].$$

Proof.

As $\mathscr{F}_{\rho\wedge\tau}\subseteq\mathscr{F}_\rho$, and $1_{\rho<\tau}$ is $\mathscr{F}_{\rho\wedge\tau}$-measurable,

$$1_{\rho\leq\tau}\mathbb{E}[X|\mathscr{F}_{\rho\wedge\tau} = \mathbb{E}[1_{\rho\leq\tau}\mathbb{E}[X|\mathscr{F}_\rho]|\mathscr{F}_{\rho\wedge\tau}].$$

Therefore, it's enough to show that $1_{\rho\leq\tau}\mathbb{E}[X|\mathscr{F}_\rho]$ is $\mathscr{F}_{\rho\wedge\tau}$-measurable. This follows from the fact that if $A\in\mathscr{F}_\rho$, then

$$A\cap\{\rho\leq\tau\}\cap\{\tau\leq t\} = (A\cap\{\rho\leq t\})\cap\{\tau\leq t\}\cap\{\rho\wedge t\leq\tau\wedge t\},$$

so $A\cap\{\rho\leq\tau\}\in\mathscr{F}_\tau$. $\square$