B8.2 Continuous Martingales and Stochastic Calculus

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Random processes

Filtrations and processes

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- We now wish to extend our thinking from random variables to random processes.
- This requires a bit of measure theoretic care to make sure everything is well defined. Recall that $\mathscr{B}(\mathbb{R})$ is the Borel σ-algebra on $\mathbb{R}$, that is, the smallest σ-algebra containing all open sets/such that all continuous functions are measurable.

Definition

The mapping $t \mapsto X_t(\omega)$ for a fixed $\omega \in \Omega$, represents a realisation of our stochastic process, called a *sample path* or *trajectory*. We shall assume that

$$(t,\omega) \mapsto X_t(\omega) : \left([0,\infty) \times \Omega, \mathscr{B}([0,\infty)) \otimes \mathscr{F}\right) \mapsto \left(\mathbb{R}, \mathscr{B}(\mathbb{R})\right)$$

is measurable (i.e. $\forall A \in \mathscr{B}(\mathbb{R})$, $\{(t,\omega) : X_t \in A\}$ is in the product σ-algebra $\mathscr{B}([0,\infty)) \otimes \mathscr{F}$). Our stochastic process is then said to be *measurable*.

In discrete time, we often made statements which held 'almost surely', that is, up to a set of measure zero. In continuous time, we need to be more careful with what this means:

Definition

Let X, Y be two stochastic processes defined on a common probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

1. We say that X is a *modification* of Y if, for all $t \geq 0$, we have $X_t = Y_t$ a.s.;
2. We say that X and Y are *indistinguishable* if

$$\mathbb{P}\big[X_t = Y_t \text{ for all } 0 \leq t < \infty\big] = 1,$$

or equivalently, $X_t(\omega) = Y_t(\omega)$ for all t, for all $\omega \notin \mathscr{N}$ where $\mathbb{P}(\mathscr{N}) = 0$.

Example

Let $T \sim U([0,1])$ be a uniform random variable, and take the random process $X_t = 1_{t=T}$. Then $Y_t := 0$ is a modification of X_t, as $Y_t = X_t$ a.s. for each t. However, Y and X are not indistinguishable, as $X \neq Y$ for *some* t with positive probability (in fact, with probability 1).

If X and Y are modifications of one another then, in particular, they have the same finite dimensional distributions,

$$\mathbb{P}[(X_{t_1},\ldots,X_{t_n})\in A]=\mathbb{P}[(Y_{t_1},\ldots,Y_{t_n})\in A]$$

for all measurable sets A, but indistinguishability is a much stronger property.