B8.2 Continuous Martingales and Stochastic Calculus

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Brownian Motion

Wiener Measure

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- We mentioned before the idea that a random process can be thought of as a random variable in the space of functions
- We will now quickly see how this can be made more precise.

- Let $C(\mathbb{R}_+, \mathbb{R})$ be the space of continuous functions from $[0, \infty)$ to $\mathbb{R}$.
- Given a Brownian motion $(B_t : t \geq 0)$ on $(\Omega, \mathscr{F}, \mathbb{P})$, consider the map

$$\Omega \to C(\mathbb{R}_+, \mathbb{R}), \quad \text{given by} \quad \omega \mapsto (B_t(\omega) : t \geq 0) \tag{1}$$

- This is measurable w.r.t. $\mathscr{B}(C(\mathbb{R}_+, \mathbb{R}))$ – the smallest σ-algebra such that the coordinate mappings (i.e. $(\omega_t : t \geq 0) \mapsto \omega(t_0)$ for a fixed t_0) are measurable.
- (In fact $\mathscr{B}(C(\mathbb{R}_+, \mathbb{R}))$ is also the Borel σ-algebra generated by the topology of uniform convergence on compacts.)

Definition

The *Wiener measure* W is the image of $\mathbb{P}$ under the mapping in (1); it is the probability measure on the space of continuous functions such that the canonical process, i.e. $(B_t(\omega) = \omega(t), t \geq 0)$, is a Brownian motion.

In other words, W is the unique probability measure on $(C(\mathbb{R}_+,\mathbb{R}),\mathscr{B}(C(\mathbb{R}_+,\mathbb{R})))$ such that

1. $\mathrm{W}(\{\omega \in C(\mathbb{R}_+,\mathbb{R}), \omega(0)=0\})=1$;
2. for any $n \geq 1$, $\forall 0=t_0<t_1<\ldots<t_n$, $A \in \mathscr{B}(\mathbb{R}^n)$

$$\begin{aligned}
&\mathrm{W}(\{\omega \in C(\mathbb{R}_+,\mathbb{R}) : (\omega(t_1),\ldots,\omega(t_n)) \in A\}) \\
&= \int_A \frac{1}{(2\pi)^{\frac{n}{2}}} \frac{dy_1\cdots dy_n}{\sqrt{t_1(t_2-t_1)\ldots(t_n-t_{n-1})}} \exp\Big(-\sum_{i=1}^n \frac{(y_i-y_{i-1})^2}{2(t_i-t_{i-1})}\Big),
\end{aligned}$$

where $y_0 := 0$.

(Uniqueness follows from the Monotone Class Lemma, since $\mathscr{B}(C(\mathbb{R}_+,\mathbb{R})))$ is generated by finite dimensional projections.)