B8.2 Continuous Martingales and Stochastic Calculus

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Random processes

Measure theory

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We seek to understand *random processes*. Formally, we can think of this in two parts:

(i) a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, describing *states of the world* $\omega \in \Omega$, *events* $A \in \mathscr{F}$ and their *probabilities* $\mathbb{P}$, and

(ii) mappings $X : (\Omega, \mathscr{F}) \to (E, \mathscr{E})$, giving the value of the random variable in each state of the world.

- We need to assume $X : \Omega \to E$ is a *measurable* mapping, so that for each $e \in \mathscr{E}$, $X^{-1}(e) \in \mathscr{F}$ and so, in particular, we can assign a probability to the event that $X \in e$.
- Often $(E, \mathscr{E})$ is just $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ (where $\mathscr{B}(\mathbb{R})$ is the Borel sets on $\mathbb{R}$) and this just says that for each $x \in \mathbb{R}$ we can assign a probability to the event $\{X \leq x\}$.

Definition

A *stochastic process*, indexed by some set $\mathscr{T}$, is a collection of random variables $\{X_t\}_{t \in \mathscr{T}}$, defined on a common probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and taking values in a common *state space* $(E, \mathscr{E})$.

- For us, $\mathscr{T}$ will generally be either $[0, \infty)$ or $[0, T]$ and we think of X_t as a random quantity that evolves with time.

Definition

A collection $\{\mathscr{F}_t, t \in [0,\infty)\}$ of σ-algebras of sets in $\mathscr{F}$ is a *filtration* if $\mathscr{F}_t \subseteq \mathscr{F}_{t+s}$ for $t, s \in [0,\infty)$. (Intuitively, $\mathscr{F}_t$ corresponds to the information known to an observer at time t.)

In particular, for a process X we define $\mathscr{F}_t^X = \sigma(\{X(s) : s \leq t\})$ (that is $\mathscr{F}_t^X$ is the information obtained by observing X up to time t) to be the *natural filtration* associated with the process X.