B8.2 Continuous Martingales and Stochastic Calculus

(Sub/super-)Martingales in continuous time

Martingale convergence and optional stopping

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We earlier showed that:

Theorem

Let X be a supermartingale with right continuous sample paths. Assume that $(X_t)_{t\geq 0}$ is bounded in L^1, i.e. $\sup_t \mathbb{E}[|X_t|] < \infty$ (or more generally $\sup_t \mathbb{E}[X_t^-] < \infty$). Then there exists $X_\infty \in L^1$ such that $\lim_{t\to\infty} X_t = X_\infty$ almost surely.

- Under the assumptions of this theorem, X_t may not converge to X_∞ in L^1.
- The next result gives, for martingales, necessary and sufficient conditions for L^1-convergence.

Definition

A martingale is said to be *closed* if there exists a random variable $Z \in L^1$ such that for every $t \geq 0$, $X_t = \mathbb{E}[Z|\mathscr{F}_t]$.

Theorem (Martingale Convergence Theorem)

Let $(X_t : t \geq 0)$ be a martingale with right continuous sample paths. Then TFAE:

1. *X is closed;*
2. *the collection $(X_t)_{t\geq 0}$ is uniformly integrable;*
3. *X_t converges almost surely and in L^1 as $t \to \infty$.*

Moreover, if these properties hold, $X_t = \mathbb{E}[X_\infty|\mathscr{F}_t]$ for every $t \geq 0$, where $X_\infty \in L^1$ is the almost sure limit of X_t as $t \to \infty$.

Proof.

That the first condition implies the second is easy. If $Z \in L^1$, then $\mathbb{E}[Z|\mathscr{G}]$, where $\mathscr{G}$ varies over sub σ-fields of $\mathscr{F}$ is uniformly integrable.

As ii implies Theorem the limit exists a.s. and is in L^1, under both ii and iii we have almost sure convergence. Vitali's theorem then states that ii and iii are equivalent.

Finally, if the third condition holds, for every $s \geq 0$, pass to the limit as $t \to \infty$ in the equality $X_s = \mathbb{E}[X_t|\mathscr{F}_s]$ (using the fact that conditional expectation is continuous for the L^1-norm, see appendix) and obtain $X_s = \mathbb{E}[X_\infty|\mathscr{F}_s]$. $\square$

- We would now like to establish conditions under which we have an optional stopping theorem for continuous martingales.
- As usual, our starting point will be the corresponding discrete time result and we shall pass to a suitable limit.

Theorem (Optional stopping for uniformly integrable discrete time martingales)

Let $(Y_n)_{n\in\mathbb{N}}$ *be a uniformly integrable martingale with respect to the filtration* $(\mathscr{G}_n)_{n\in\mathbb{N}}$, *and let* Y_∞ *be the a.s. limit of* Y_n *when* $n\to\infty$. *Then, for every choice of the stopping times* S *and* T *such that* $S\le T$, *we have* $Y_T\in L^1$ *and*

$$Y_S=\mathbb{E}[Y_T|\mathscr{G}_S],$$

where

$$\mathscr{G}_S=\{A\in\mathscr{G}_\infty : A\cap\{S=n\}\in\mathscr{G}_n \text{ for every } n\in\mathbb{N}\},$$

with the convention that $Y_T=Y_\infty$ *on the event* $\{T=\infty\}$, *and similarly for* Y_S.

Let $(X_t)_{t\geq 0}$ be a right continuous martingale or supermartingale such that X_t converges almost surely as $t\to\infty$ to a limit X_∞. Then for every stopping time T, we define

$$X_T(\omega)=1_{\{T(\omega)<\infty\}}X_{T(\omega)}(\omega)+1_{\{T(\omega)=\infty\}}X_\infty(\omega).$$

Theorem

Let $(X_t)_{t\geq 0}$ be a uniformly integrable martingale with right continuous sample paths. Let S and T be two stopping times with $S\leq T$. Then X_S and X_T are in L^1 and $X_S=\mathbb{E}[X_T|\mathscr{F}_S]$.
In particular, for every stopping time S we have $X_S=\mathbb{E}[X_\infty|\mathscr{F}_S]$ and $\mathbb{E}[X_S]=\mathbb{E}[X_\infty]=\mathbb{E}[X_0]$.

Proof

For any integer $n \geq 0$ set

$$T_n = \sum_{k=0}^{\infty} \frac{k+1}{2^n} 1_{\{k2^{-n} < T \leq (k+1)2^{-n}\}} + \infty 1_{\{T=\infty\}},$$

$$S_n = \sum_{k=0}^{\infty} \frac{k+1}{2^n} 1_{\{k2^{-n} < S \leq (k+1)2^{-n}\}} + \infty 1_{\{S=\infty\}}.$$

Then T_n and S_n are sequences of stopping times that decrease respectively to T and S. Moreover, $S_n \leq T_n$ for every $n \geq 0$.

For each fixed n, $2^n S_n$ and $2^n T_n$ are stopping times of the discrete filtration $\mathscr{G}_n = \mathscr{F}_{k/2^n}$ and $Y_k^{(n)} = X_{k/2^n}$ is a discrete martingale with respect to this filtration.

Discrete optional stopping implies $Y^{(n)}_{2^n S_n}, Y^{(n)}_{2^n T_n}$ are in L^1 and

$$X_{S_n} = Y^{(n)}_{2^n S_n} = \mathbb{E}[Y^{(n)}_{2^n T_n} | \mathscr{G}_{2^n S_n}] = \mathbb{E}[X_{T_n} | \mathscr{F}_{S_n}].$$

Let $A \in \mathscr{F}_S$. Since $\mathscr{F}_S \subseteq \mathscr{F}_{S_n}$ we have $A \in \mathscr{F}_{S_n}$ and so $\mathbb{E}[1_A X_{S_n}] = \mathbb{E}[1_A X_{T_n}]$.

By right continuity, $X_S = \lim_{n\to\infty} X_{S_n}$ and $X_T = \lim_{n\to\infty} X_{T_n}$. The limits also hold in L^1 (in fact, by optional stopping, $X_{S_n} = \mathbb{E}[X_\infty | \mathscr{F}_{S_n}]$ for every n and so $(X_{S_n})_{n\geq 1}$ and $(X_{T_n})_{n\geq 1}$ are uniformly integrable).

L^1 convergence implies that the limits X_S and X_T are in L^1 and allows us to pass to a limit, $\mathbb{E}[1_A X_S] = \mathbb{E}[1_A X_T]$. This holds for all $A \in \mathscr{F}_S$ and so since X_S is $\mathscr{F}_S$-measurable we conclude that $X_S = \mathbb{E}[X_T | \mathscr{F}_S]$, as required. $\square$

Corollary

In particular, for any martingale with right continuous paths and two bounded *stopping times, $S \leq T$, we have $X_S, X_T \in L^1$ and $X_S = \mathbb{E}[X_T | \mathscr{F}_S]$.*

Proof.

Let a be such that $S \leq T \leq a$. The martingale $(X_{t\wedge a})_{t\geq 0}$ is closed by X_a and so we may apply our previous results. $\square$

Corollary

Suppose that $(X_t)_{t\geq 0}$ *is a martingale with right continuous paths and* T *is a stopping time.*

1. $X^T = (X_{t\wedge T})_{t\geq 0}$ *is a martingale;*
2. *if, in addition,* $(X_t)_{t\geq 0}$ *is uniformly integrable, then* $X^T = (X_{t\wedge T})_{t\geq 0}$ *is uniformly integrable and for every* $t \geq 0$, $X_{t\wedge T} = \mathbb{E}[X_T|\mathscr{F}_t]$.

Proof.

We know $X_t^T = X_{t\wedge T} = X_T^t$, and that X_t is integrable. Hence, by the optional stopping theorem applied to the stopped process X^t, we see that X_t^T is integrable for every t. Furthermore, for any $s < t$, as $T \wedge s$ and $T \wedge t$ are bounded stopping times, by the optional stopping theorem and properties of $\mathscr{F}_{T\wedge s}$ (Lemma 4.12),

$$\begin{aligned} X_s^T &= X_{T\wedge s} = \mathbb{E}[X_{T\wedge t}|\mathscr{F}_{T\wedge s}] \\ &= 1_{T<s}X_T + 1_{T\geq s}\mathbb{E}[X_{T\wedge t}|\mathscr{F}_{T\wedge s}] \\ &= 1_{T<s}X_{T\wedge t} + 1_{T\geq s}\mathbb{E}[X_{T\wedge t}|\mathscr{F}_s] = \mathbb{E}[X_t^T|\mathscr{F}_s]. \end{aligned}$$

Therefore X^T is a martingale. □

A converse result is also possible:

Theorem

Suppose M is a right-continuous process defined for $t < \infty$, and adapted to a right-continuous filtration $\{\mathscr{F}_t\}_{t<\infty}$. Then M is a martingale if, and only if, for every bounded stopping time T we know $\mathbb{E}\big[|M_T|\big] < \infty$ and $\mathbb{E}[M_T] = \mathbb{E}[M_0]$.

Proof.

By considering the process $\{M_t - M_0\}_{t\geq 0}$, we can assume without loss of generality that $E[M_T] = E[M_0] = 0$. If M is a martingale, then $M_t = E[M_T|\mathscr{F}_t]$, and the result follows by optional stopping and Jensen's inequality.

Conversely, consider any times $s < t \in [0,\infty)$ and any $A \in \mathbb{F}_s$. Define a random time T by putting $T(\omega) = s$ if $\omega \in A$ and $T(\omega) = t$ if $\omega \notin A$. Then T is a stopping time. By hypothesis

$$\mathbb{E}[1_A M_s] + \mathbb{E}[1_{A^c} M_t] = \mathbb{E}[M_T] = 0 = \mathbb{E}[M_t] = \mathbb{E}[1_A M_t] + \mathbb{E}[1_{A^c} M_t].$$

Therefore

$$\mathbb{E}[1_A M_s] = \mathbb{E}[1_A M_t]$$

for all $A \in \mathbb{F}_s$, so $M_s = E[M_t|\mathscr{F}_s]$ almost surely. $\square$

Above all, optional stopping is a powerful tool for calculations.

Example

Fix $a > 0$ and let T_a be the first hitting time of a by standard Brownian motion. Then for each $\lambda > 0$,

$$\mathbb{E}[e^{-\lambda T_a}] = e^{-a\sqrt{2\lambda}}.$$

Recall that $N_t^\lambda = \exp(\lambda B_t - \frac{\lambda^2}{2}t)$ is a martingale. So $N_{t\wedge T_a}^\lambda$ is still a martingale and it is in the bounded interval $[0, e^{\lambda a}]$ and hence is uniformly integrable, so $\mathbb{E}[N_{T_a}^\lambda] = \mathbb{E}[N_0^\lambda]$. That is,

$$e^{a\lambda}\mathbb{E}[e^{-\lambda^2 T_a/2}] = \mathbb{E}[N_0^\lambda] = 1.$$

Replace λ by $\sqrt{2\lambda}$ and rearrange.
Warning: This argument fails if $\lambda < 0$.