B8.2 Continuous Martingales and Stochastic Calculus

Stochastic Integration

Intrinsic characterisation of stochastic integrals

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- We can also characterise the Itô integral in a more abstract way.
- This is less intuitive than the last method (as it's not constructive), but gives a particularly useful way to analyse the properties of the integral.

Theorem

Let $M \in \mathscr{H}^{2,c}$. *For any* $K \in \mathrm{L}^2(M)$ *there exists a unique element in* $\mathscr{H}_0^{2,c}$, *denoted* $K \bullet M$, *such that*

$$\langle K \bullet M, N \rangle = K \cdot \langle M, N \rangle, \quad \forall N \in \mathscr{H}^{2,c}. \tag{1}$$

Furthermore, $\|K \bullet M\|_{\mathscr{H}^{2,c}} = \|K\|_{\mathrm{L}^2(M)}$ *and the map*

$$K \mapsto K \bullet M$$
$$L^2(M) \to \mathscr{H}_0^{2,c}$$

is a linear isometry.

Proof

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We first check uniqueness. Suppose that there are two such elements, X and X'. Then

$$\langle X, N\rangle - \langle X', N\rangle = \langle X - X', N\rangle \equiv 0, \quad \forall N \in \mathscr{H}^{2,c}.$$

Taking $N = X - X'$, as the only martingale with zero quadratic variation is identically zero, we conclude that $X = X'$.

Now let us verify $\langle K \bullet M, N \rangle = K \cdot \langle M, N \rangle$ for the Itô integral.

Fix $N \in \mathscr{H}^{2,c}$. First note that for $K \in L^2(M)$ the Kunita–Watanabe inequality shows that

$$\mathbb{E}\left[\int_0^\infty |K_s||d\langle M, N\rangle_s|\right] \leq \|K\|_{L^2(M)} \|N\|_{\mathscr{H}^{2,c}} < \infty$$

and thus the variable

$$\int_0^\infty K_s d\langle M, N\rangle_s = \Big(K \cdot \langle M, N\rangle\Big)_\infty$$

is well defined and in L^1.

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If K is an elementary process, writing $K = \sum K^{(i)} 1_{(t_i, t_{i+1}]}$, and $M^i_s = K^{(i)}(M_{s \wedge t_{i+1}} - M_{s \wedge t_i})$ we have

$$\langle K \bullet M, N \rangle = \sum_{i=0}^{m} \langle M^i, N \rangle$$

and

$$\langle M^i, N \rangle_t = K^{(i)} \Big(\langle M, N \rangle_{t_{i+1} \wedge t} - \langle M, N \rangle_{t_i \wedge t} \Big),$$

so

$$\langle K \bullet M, N \rangle_t = \sum K^{(i)} \Big(\langle M, N \rangle_{t_{i+1} \wedge t} - \langle M, N \rangle_{t_i \wedge t} \Big) = \int_0^t K_s d \langle M, N \rangle_s.$$

Now observe that the mapping $X \mapsto \langle X, N\rangle_\infty$ is continuous from $\mathscr{H}^{2,c}$ into L^1. Indeed, by Kunita–Watanabe

$$\mathbb{E}\Big[\big|\langle X, N\rangle\big|\Big] \le \mathbb{E}\Big[\langle X, X\rangle_\infty\Big]^{1/2} \mathbb{E}\Big[\langle N, N\rangle_\infty\Big]^{1/2} = \|N\|_{\mathscr{H}^{2,c}} \|X\|_{\mathscr{H}^{2,c}}.$$

So if K^n is a sequence in $\mathscr{E}$ such that $K^n \to K$ in $L^2(M)$,

$$\langle K \bullet M, N\rangle_\infty = \lim_{n\to\infty} \langle K^n \bullet M, N\rangle_\infty = \lim_{n\to\infty} \Big(K^n \cdot \langle M, N\rangle\Big)_\infty = \Big(K \cdot \langle M, N\rangle\Big)_\infty,$$

where the convergence is in L^1 and the last equality is again a consequence of Kunita–Watanabe by writing

$$\mathbb{E}\left[\left|\int_0^\infty \Big(K_s^n - K_s\Big) d\langle M, N\rangle_s\right|\right] \le \mathbb{E}\Big[\langle N, N\rangle_\infty\Big]^{1/2} \|K^n - K\|_{L^2(M)}.$$

We have thus obtained

$$\langle K \bullet M, N \rangle_\infty = \Big(K \cdot \langle M, N \rangle\Big)_\infty,$$

but replacing N by the stopped martingale N^t in this identity also gives

$$\langle K \bullet M, N \rangle_t = \Big(K \cdot \langle M, N \rangle\Big)_t.$$

We have therefore shown that this property uniquely identifies the integral.

That it is linear (in the integrand) and an isometry (from $L^2(M)$ to $\mathscr{H}^2$) comes from the earlier construction that we gave. $\square$