B8.2 Continuous Martingales and Stochastic Calculus

Continuous semimartingales

Quadratic covariation

Samuel Cohen

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UNIVERSITY OF OXFORD
Mathematical Institute

Oxford Mathematics

- We can see that the quadratic variation of a martingale is telling us something about how its variance increases with time.
- We also need an analogous quantity for the 'covariance' between two martingales.
- This is most easily defined through *polarisation*.

Definition

The quadratic *co-variation* between two continuous local martingales M, N is defined by

$$\langle M, N\rangle := \frac{1}{2}\left(\langle M+N, M+N\rangle - \langle M, M\rangle - \langle N, N\rangle\right). \qquad (1)$$

It is often called the (angle) bracket process of M and N.

Proposition

For two continuous local martingales M, N

1. *the process $\langle M, N\rangle$ is the unique finite variation process, zero at zero, such that $(M_t N_t - \langle M, N\rangle_t : t \geq 0)$ is a continuous local martingale;*
2. *the mapping $M, N \mapsto \langle M, N\rangle$ is bilinear and symmetric;*
3. *for any stopping time τ,*

$$\langle M^\tau, N^\tau\rangle_t = \langle M^\tau, N\rangle_t = \langle M, N^\tau\rangle_t = \langle M, N\rangle_{\tau\wedge t}, \quad t \geq 0, \text{ a.s.}; \tag{2}$$

4. *for any $t > 0$ and a sequence of partitions π_n of $[0, t]$ with mesh converging to zero*

$$\sum_{t_i\in\pi_n} (M_{t_{i+1}} - M_{t_i})(N_{t_{i+1}} - N_{t_i}) \to \langle M, N\rangle_t, \tag{3}$$

the convergence being in probability.

Proof

(i) $(M+N)_t^2 - \langle M+N, M+N\rangle_t$ is a continuous local martingale and by adding and subtracting terms it is equal to

$$\underbrace{M_t^2 - \langle M, M\rangle_t}_{\text{l.mat}} + \underbrace{N_t^2 - \langle N, N\rangle_t}_{\text{l.mat}}$$
$$+2\underbrace{\left(M_tN_t - \frac{1}{2}(\langle M+N, M+N\rangle_t - \langle M, M\rangle_t - \langle N, N\rangle_t)\right)}_{\text{hence also a l.mat}}$$

Uniqueness follows from triviality of continuous finite variation martingales.

(iv) Note that

$$(M_t + N_t - M_s - N_s)^2 - (M_t - M_s)^2 - (N_t - N_s)^2 = 2(M_t - M_s)(N_t - N_s).$$

The asserted convergence then follows from the definition of the quadratic variation.

(ii) Both properties follow from (iv). Symmetry is obvious from the definition in (1).

(iii) Follows from (iv). $\square$

Definition

Two continuous local martingales M, N, are said to be *(very strongly) orthogonal* if $\langle M, N \rangle = 0$.

For example, if B and B' are independent Brownian motions, then $\langle B, B' \rangle = 0$.

Remark

It follows that if M and N are two martingales bounded in L^2 and with $M_0 N_0 = 0$ a.s., then $(M_t N_t - \langle M, N \rangle_t, t \geq 0)$ is a uniformly integrable martingale. In particular, for every stopping time τ,

$$\mathbb{E}[M_\tau N_\tau] = \mathbb{E}[\langle M, N \rangle_\tau]. \tag{4}$$

Remark

Note that $\langle M, N \rangle = 0$ is a stronger statement than $\mathbb{E}[M_\infty N_\infty] = \mathbb{E}[\langle M, N \rangle_\infty] = 0$. For example, consider W a Brownian motion and $N = \xi W$, for ξ independent of W with mean zero ($\mathscr{F}_0$-measurable). Then $\langle W, N \rangle = \xi t \neq 0$, but $\mathbb{E}[\langle W, N \rangle_\infty] = 0$.

Take $M, N \in \mathscr{H}_0^{2,c}$, which we recall had the norm $\|M\|^2_{\mathscr{H}^{2,c}} = \mathbb{E}[\langle M, M\rangle_\infty] = \mathbb{E}[M_\infty^2]$.

This norm is consistent with the inner product on $\mathscr{H}^{2,c} \times \mathscr{H}^{2,c}$ given by $\mathbb{E}[M_\infty N_\infty]$ and, by the usual Cauchy–Schwarz inequality,

$$\begin{aligned}\mathbb{E}[\langle M, N\rangle_\infty] = \mathbb{E}[M_\infty N_\infty] &\leq \mathbb{E}[|M_\infty N_\infty|] \\ &\leq \sqrt{\mathbb{E}[M_\infty^2]\mathbb{E}[N_\infty^2]} = \sqrt{\mathbb{E}[\langle M\rangle_\infty]\mathbb{E}[\langle N\rangle_\infty]}.\end{aligned}$$

It is easy to obtain an almost sure result also, using that

$$\left|\sum\left(M_{t_{i+1}}-M_{t_i}\right)\left(N_{t_{i+1}}-N_{t_i}\right)\right| \leq \sqrt{\sum\left(M_{t_{i+1}}-M_{t_i}\right)^2}\sqrt{\sum\left(N_{t_{i+1}}-N_{t_i}\right)^2}$$

and taking limits to deduce that

$$|\langle M, N\rangle_t| \leq \sqrt{\langle M\rangle_t}\sqrt{\langle N\rangle_t}.$$

It's often convenient to have a more general version of this inequality.

Theorem (Kunita–Watanabe inequality)

Let M, N be continuous local martingales and K, H two measurable processes. Then for all $0 \le t \le \infty$,

$$\int_0^t |H_s||K_s||\mathrm{d}\langle M, N\rangle_s| \le \left(\int_0^t H_s^2 \mathrm{d}\langle M\rangle_s\right)^{1/2} \left(\int_0^t K_s^2 \mathrm{d}\langle N\rangle_s\right)^{1/2} \ a.s.. \tag{5}$$

We omit the proof which approximates H, K by simple functions and then essentially uses the Cauchy–Schwarz inequality for sums noted above.