UNIVERSITY OF OXFORD

Mathematical Institute

B8.2 Continuous Martingales and Stochastic Calculus

Brownian Motion

Properties of Brownian motion

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- Although the sample paths of Brownian motion are continuous, it does not mean that they are nice in any other sense.
- In fact the behaviour of Brownian motion is distinctly odd.
- Here are just a few of its strange behavioural traits.

1. Although $\{B_t\}_{t \geq 0}$ is continuous everywhere, it is (with probability one) differentiable nowhere.
2. Brownian motion will eventually hit any and every real value no matter how large, or how negative. No matter how far above the axis, it will (with probability one) be back down to zero at some later time.
3. Once Brownian motion hits a value, it immediately hits it again (uncountably!) *infinitely* often, and then again from time to time in the future.
4. It doesn't matter what scale you examine Brownian motion on, it looks just the same. Brownian motion is a fractal process.

The last property is really a consequence of the construction of the process. We'll formulate the second and third more carefully later.

Proposition

Let B be a standard real-valued Brownian motion. Then

1. $-B_t$ *is also a Brownian motion,* *(symmetry)*
2. $\forall\, c \geq 0$, cB_{t/c^2} *is a Brownian motion,* *(scaling)*
3. $X_0 = 0$, $X_t := tB_{\frac{1}{t}}$ *is a Brownian motion,* *(time reversal)*
4. $\forall\, s \geq 0$, $\tilde{B}_t = B_{t+s} - B_s$ *is a Brownian motion independent of* $\sigma(B_u : u \leq s)$, *(simple Markov property).*

The proof is an exercise.