UNIVERSITY OF OXFORD
Mathematical Institute

B8.2 Continuous Martingales and Stochastic Calculus

Stochastic Integration

Stochastic integral w.r.t. L^2-bounded martingales

Samuel Cohen
Hilary Term 2021

Oxford Mathematics

- At the beginning of the course we argued that whereas classically differential equations take the form

$$dX(t) = a(t, X(t))dt,$$

in many settings, the dynamics of interest may also have a random component and so perhaps take the form

$$dX_t = a(t, X_t)dt + b(t, X_t)dB_t.$$

- We actually understand equations like this in the integral form:

$$X_t - X_0 = \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dB_s.$$

- If a is nice enough, then the first term has a classical interpretation. It is the second term, or rather a generalisation of it, that we want to make sense of now.

We will take two approaches to constructing this stochastic integral:

- The first approach (this lecture) will be to mimic what we usually do for construction of the Lebesgue integral, namely work out how to integrate simple functions and then extend to general functions through passage to the limit.
- We'll then (next lecture) provide a very slick, but not at all intuitive, approach that nonetheless gives us some 'quick wins' in proving properties of the integral.

Remark on Notation:

- We are going to use the notation $\varphi \bullet M$ for the (Itô) stochastic integral of φ with respect to M.
- This is not universally accepted notation; many authors would write $\int_0^t \varphi_s dM_s$ for $(\varphi \bullet M)_t$.
- Moreover, for emphasis, when the integrator is stochastic, we have used '$\bullet$' in place of the '$\cdot$' that we used for the Stieltjes integral.

- We're going to develop a theory of integration w.r.t. martingales in $\mathscr{H}^{2,c}$.
- Recall that $\mathscr{H}_0^{2,c}$ is the space of continuous martingales M, zero at zero, which are bounded in L^2.
- It is a Hilbert space with the inner product $\langle M, N\rangle_{\mathscr{H}^{2,c}} = \mathbb{E}[M_\infty N_\infty]$ and induced norm

$$\|M\|_{\mathscr{H}^{2,c}} = \sqrt{\mathbb{E}[M_\infty^2]} = \sqrt{\mathbb{E}[\langle M\rangle_\infty]}.$$

(In a very real sense we are identifying $\mathscr{H}^{2,c}$ with L^2.)

Step 1: Riemann sums and simple integrals

- Define $\mathscr{E}$ to be the space of simple bounded process of the form

$$\varphi_t = \sum_{i=0}^{m} \varphi^{(i)} 1_{(t_i, t_{i+1}]}(t), \quad t \geq 0, \tag{1}$$

for some $m \in \mathbb{N}$, $0 \leq t_0 < t_1 < \ldots < t_{m+1}$ and where $\varphi^{(i)}$ are bounded $\mathscr{F}_{t_i}$-measurable random variables.

- Define the stochastic integral $\varphi \bullet M$ of φ in (1) with respect to $M \in \mathscr{H}^{2,c}$ via

$$(\varphi \bullet M)_t := \sum_{i=0}^{m} \varphi^{(i)} (M_{t \wedge t_{i+1}} - M_{t \wedge t_i}), \quad t \geq 0. \tag{2}$$

- If we write $M^i_t := \varphi^{(i)}(M_{t\wedge t_{i+1}} - M_{t\wedge t_i})$ then clearly $M^i \in \mathscr{H}^{2,c}$ and so $\varphi \bullet M$ is a martingale.
- Moreover, since for $i \neq j$ the intervals $(t_i, t_{i+1}]$ and $(t_j, t_{j+1}]$ are disjoint, $M^i_t M^j_t$ is a martingale and hence $\langle M^i, M^j\rangle_t = 0$.
- Using the bilinearity of the bracket process then yields

$$\begin{aligned}\langle \varphi \bullet M\rangle_t = \sum_{i=0}^{m} \langle M^i\rangle_t = \sum_{i=0}^{m} \left(\varphi^{(i)}\right)^2 \left(\langle M\rangle_{t_{i+1}\wedge t} - \langle M\rangle_{t_i\wedge t}\right) \\ = \int_0^t \varphi_s^2 \mathrm{d}\langle M\rangle_s, \quad t \geq 0.\end{aligned}$$

- We already used the notation that if K is progressively measurable and A is of finite variation, then

$$(K \cdot A)_t = \int_0^t K_s(\omega) dA_s(\omega), \qquad t \geq 0.$$

- In that notation

$$\langle \varphi \bullet M \rangle = \varphi^2 \cdot \langle M \rangle.$$

More generally, for $N \in \mathscr{H}^{2,c}$,

$$\begin{aligned} \langle \varphi \bullet M, N \rangle_t = \sum_{i=0}^{m} \langle M^i, N \rangle_t = \sum_{i=0}^{m} \varphi^{(i)} \left(\langle M, N \rangle_{t_{i+1} \wedge t} - \langle M, N \rangle_{t_i \wedge t} \right) \\ = \int_0^t \varphi_s \mathrm{d} \langle M, N \rangle_s = (\varphi \cdot \langle M, N \rangle)_t \, . \end{aligned} \tag{3}$$

Proposition

Let $M \in \mathscr{H}^{2,c}$. The mapping $\varphi \mapsto \varphi \bullet M$ is a linear map from $\mathscr{E}$ to $\mathscr{H}_0^{2,c}$. Moreover,

$$\|\varphi \bullet M\|^2_{\mathscr{H}^{2,c}} = \mathbb{E}\left[\int_0^\infty \varphi_t^2 \mathrm{d}\langle M\rangle_t\right]. \tag{4}$$

- The proof is easy – we just need to show linearity.
- But given φ, $\psi \in \mathscr{E}$, we use a refinement of the partitions on which they are constant to write them as simple functions with respect to the same partition and the result is trivial.

Remark

If we were considering martingales with jumps, then it would be important that the processes in $\mathscr{E}$ are left continuous.

Step 2: Simple integrals are dense

- We are expecting an L^2-theory – we have already found an expression for the 'L^2-norm' of $\varphi \bullet M$.
- Let us define the appropriate spaces more carefully.

Definition

Given $M \in \mathscr{H}^{2,c}$ we denote by $\mathrm{L}^2(M)$ the space of progressively measurable processes K such that

$$\|K\|^2_{\mathrm{L}^2(M)} := \mathbb{E}\left[\int_0^\infty K_t^2 \mathrm{d}\langle M\rangle_t\right] < +\infty. \tag{5}$$

- $L^2(M)$ is a Hilbert space, with inner product

$$H, K \mapsto \mathbb{E}\left[\int_0^\infty H_t K_t d\langle M\rangle_t\right] = \mathbb{E}\left[(HK \cdot \langle M\rangle)_\infty\right].$$

- We have $\mathscr{E} \subseteq L^2(M)$ and our definition tells us that the map $\mathscr{E} \to \mathrm{H}_0^2$ given by $\varphi \mapsto \varphi \bullet M$ is a linear isometry.
- If we can show that the elementary functions are dense in $L^2(M)$, this observation will allow us to define integrals of functions from $L^2(M)$ with respect to M via a limiting procedure.

Proposition

Let $M \in \mathscr{H}^{2,c}$. Then $\mathscr{E}$ is a dense vector subspace of $\mathrm{L}^2(M)$.

Practically, this means that any function in $\mathrm{L}^2(M)$ can be approximated (in $\mathrm{L}^2(M)$-norm) by a sequence of simple functions.

Proof

UNIVERSITY OF OXFORD
Mathematical Institute

It is enough to show that if $K \in \mathrm{L}^2(M)$ is orthogonal to φ for all $\varphi \in \mathscr{E}$, then $K = 0$ (as an element of $\mathrm{L}^2(M)$).

So suppose that $\langle K, \varphi \rangle_{\mathrm{L}^2(M)} = 0$ for all $\varphi \in \mathscr{E}$. Let $X = K \cdot \langle M \rangle$, i.e. $X_t = \int_0^t K_u \mathrm{d}\langle M \rangle_u$.

This is well defined and, by Cauchy–Schwarz/Kunita–Watanabe

$$\mathbb{E}[|X_t|] \leq \mathbb{E}\left[\int_0^t |K_u| \mathrm{d}\langle M \rangle_u\right] \leq \sqrt{\mathbb{E}\left[\int_0^t K_u^2 \mathrm{d}\langle M \rangle_u\right]} \sqrt{\mathbb{E}\langle M \rangle_t} < +\infty$$

since $M \in \mathscr{H}^{2,c}$ and $K \in \mathrm{L}^2(M)$ (we took one of the functions to be identically one in Cauchy–Schwarz).

Taking $\varphi = \xi 1_{(s,t]} \in \mathscr{E}$, with $0 \leq s < t$ and ξ a bounded $\mathscr{F}_s$-measurable r.v., we have

$$0 = \langle K, \varphi \rangle_{\mathrm{L}^2(M)} = \mathbb{E}\left[\xi \int_s^t K_u \mathrm{d}\langle M \rangle_u\right] = \mathbb{E}\left[\xi(X_t - X_s)\right].$$

Since this holds for any $\mathscr{F}_s$-measurable bounded ξ, we conclude that $\mathbb{E}[(X_t - X_s)|\mathscr{F}_s] = 0$.

In other words, X is a martingale. But X is also continuous and of finite variation and hence $X \equiv 0$ a.s.

Thus $K = 0 \ \mathrm{d}\langle M \rangle - a.e.\ a.s.$ and hence $K = 0$ in $\mathrm{L}^2(M)$. $\square$

Step 3: General integrals in $L^2(M)$

- We now know that any $K \in L^2(M)$ is a limit of simple processes $\varphi^n \to K$.
- For each φ^n we can define the stochastic integral $\varphi^n \bullet M$.
- The isometry property then shows that $\{\varphi^n \bullet M\}_{n\in\mathbb{N}}$ converges in $\mathscr{H}^{2,c}$ to some element that we denote $K \bullet M$ and which does not depend on the choice of approximating sequence φ^n.

Theorem

Let $M \in \mathscr{H}^{2,c}$*. The mapping* $\varphi \mapsto \varphi \bullet M$ *from* $\mathscr{E}$ *to* $\mathscr{H}_0^{2,c}$ *defined in* (2) *has a unique extension to a linear isometry from* $\mathrm{L}^2(M)$ *to* $\mathscr{H}_0^{2,c}$ *which we denote* $K \mapsto K \bullet M$*.*

Remark

For $K \in \mathrm{L}^2(M)$, the martingale $K \bullet M$ is called the Itô stochastic integral of K with respect to M and is often written as $(K \bullet M)_t = \int_0^t K_u \mathrm{d}M_u$. The isometry property may be then written as

$$\|K \bullet M\|^2_{\mathscr{H}^{2,c}} = \mathbb{E}\left[\left(\int_0^\infty K_t \mathrm{d}M_t\right)^2\right] = \mathbb{E}\left[\int_0^\infty K_t^2 \mathrm{d}\langle M\rangle_t\right] = \|K\|^2_{\mathrm{L}^2(M)}. \tag{6}$$

Mathematical Institute

Example

Let $M = B^T$ be a standard Brownian motion stopped at a time $T > 0$. Consider $K_t = t^3$. To define $K \bullet B$, we take a sequence of simple functions K^n which converge to K in the sense

$$\mathbb{E}\Big[\int_0^\infty (K_t^n - K_t)^2 d\langle M\rangle_t\Big] = \mathbb{E}\Big[\int_0^T (K_t^n - t^3)^2 dt\Big] \to 0.$$

Then the approximate integrals $K^n \bullet M$ (which are defined by Riemann sums) converge in $\mathscr{H}^2$ to a process which we call $K \bullet M$, or equivalently

$$(K \bullet M)_t = \int_0^t s^3 dB_s, \qquad \text{for } t \leq T.$$

Notice that if B is standard Brownian motion and we calculate $(B \bullet B)_t$, then

$$(B \bullet B)_t = \lim_{\|\pi\| \to 0} \sum_{j=0}^{N(\pi)-1} B_{t_j} \left(B_{t_{j+1}} - B_{t_j} \right). \tag{7}$$

We also know already that the quadratic variation is

$$t = \lim_{\|\pi\| \to 0} \sum_{j=0}^{N(\pi)-1} \left(B_{t_{j+1}} - B_{t_j} \right)^2 = B_t^2 - B_0^2 - 2 \sum_{j=0}^{N(\pi)-1} B_{t_j} \left(B_{t_{j+1}} - B_{t_j} \right),$$

and so rearranging we find

$$\int_0^t B_s dB_s = \frac{1}{2} \left(B_t^2 - B_0^2 - t \right) = \frac{1}{2} \left(B_t^2 - t \right).$$

This is *not* what one would have predicted from classical integration theory (the extra term here comes from the quadratic variation).

Even more strangely, it *matters* that in (7) we took the *left* endpoint of the interval for evaluating the integrand. On the problem sheet, you are asked to evaluate

$$\lim_{\|\pi\|\to 0} \sum B_{t_{j+1}} \left(B_{t_{j+1}} - B_{t_j} \right), \quad \text{and} \quad \lim_{\|\pi\|\to 0} \sum \frac{B_{t_j} + B_{t_{j+1}}}{2} \left(B_{t_{j+1}} - B_{t_j} \right).$$

Each gives a different answer.

We can more generally define

$$\int_0^T f(B_s) \circ dB_s = \lim_{\|\pi\| \to 0} \sum \left(\frac{f(B_{t_j}) + f(B_{t_{j+1}})}{2} \right) \left(B_{t_{j+1}} - B_{t_j} \right).$$

This is the so-called *Stratonovich integral*, and has the advantage that from the point of view of calculations, the rules of Newtonian calculus hold true.

From a modelling perspective however, it can be the wrong choice. For example, suppose that we are modelling the change in a population size over time and we use $[t_i, t_{i+1})$ to represent the $(i+1)$st generation. The change over (t_i, t_{i+1}) will be driven by the number of *adults*, so the population size at the *beginning* of the interval.