B8.2 Continuous Martingales and Stochastic Calculus

Continuous semimartingales

Continuous local martingales

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We now want to extend our integration theory to processes which are *not* of finite variation. The processes that make our theory work are slight generalisations of martingales.

Definition

An adapted process $(M_t : t \geq 0)$ is called a continuous *local* martingale if $M_0 = 0$, it has continuous trajectories a.s. and if there exists a non-decreasing sequence of stopping times $(\tau_n)_{n\geq 1}$ such that $\tau_n \uparrow \infty$ a.s. and for each n, $M^{\tau_n} = (M_{t\wedge\tau_n} : t \geq 0)$ is a (wlog uniformly integrable) martingale. We say (τ_n) *reduces* or *localizes* M.

More generally, when we do not assume that $M_0 = 0$, we say that M is a continuous local martingale if $N_t = M_t - M_0$ is a continuous local martingale.

Any martingale is a local martingale, but the converse is false.

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Example

Let ξ be a random variable not in L^1, and Z be an independent Bernoulli random variable with $p = 1/2$. Define a filtration

$$\mathscr{F}_t = \begin{cases} \{\emptyset, \Omega\} & t < 1 \\ \sigma(\xi) & t \in [1,2) \\ \sigma(\xi, Z) & t \geq 2 \end{cases}$$

and a process

$$X_t = \begin{cases} 0 & t < 2 \\ \xi Z & t \geq 2 \end{cases}$$

By taking the stopping times $\tau_n = n 1_{\{|\xi| < n\}}$, we see that X is a local martingale, but cannot be a martingale as $E[|X_2|] \not< \infty$.

Example

Let B be a Brownian motion, and ξ an independent nonnegative random variable not in L^1. Then define $X_t = B_{\xi^2 t}$, in the filtration $\{\mathscr{F}^X_{t+}\}_{t\geq 0}$. Then

$$\mathbb{E}[|X_t|] = \mathbb{E}[|B_{\xi^2 t}|] = \mathbb{E}\Big[\mathbb{E}\Big[|B_{\xi^2 t}|\Big|\xi\Big]\Big] = \sqrt{2t/\pi}\mathbb{E}[\xi] = \infty$$

so X is not a martingale. However, ξ is $\mathscr{F}^X_0$-measurable (we will see this from the fact $\langle X\rangle_t = \xi^2 t$ and right-continuity), so we can use the stopping times $\tau_n = n1_{\{\xi<n\}}$ to localize and hence verify X^{τ_n} is a martingale. As X is continuous, we can also localize with $\tau_n = \inf\{t : |X_t| \geq n\}$.

More clever examples (including where $\{X_t\}_{t\in\mathbb{R}}$ is uniformly integrable) are possible.

Proposition

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1. *A non-negative continuous local martingale such that $M_0 \in L^1$ is a supermartingale.*
2. *A continuous local martingale M such that there exists a random variable $Z \in L^1$ with $|M_t| \leq Z$ for every $t \geq 0$ is a uniformly integrable martingale.*
3. *If M is a continuous local martingale and $M_0 = 0$ (or more generally $M_0 \in L^1$), the sequence of stopping times*

$$T_n = \inf\{t \geq 0 : |M_t| \geq n\}$$

   reduces M.
4. *If M is a continuous local martingale, then for any stopping time ρ, the stopped process M^ρ is also a continuous local martingale.*

Proof

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(i) Write $M_t = M_0 + N_t$. By definition, there exists a sequence T_n of stopping times that reduces N. Thus, if $s \le t$, for every n,

$$N_{s\wedge T_n} = \mathbb{E}[N_{t\wedge T_n} | \mathscr{F}_s].$$

We can add M_0 to both sides (M_0 is $\mathscr{F}_s$-measurable and in L^1) and we find

$$M_{s\wedge T_n} = \mathbb{E}[M_{t\wedge T_n} | \mathscr{F}_s].$$

Since M takes non-negative values, let $n \to \infty$ and apply Fatou's lemma for conditional expectations to find

$$M_s \ge \mathbb{E}[M_t | \mathscr{F}_s]. \tag{1}$$

Taking $s = 0$, $\mathbb{E}[M_t] \le \mathbb{E}[M_0] < \infty$. So $M_t \in L^1$ for every $t \ge 0$, and (1) says that M is a supermartingale.

(ii) By the same argument,

$$M_{s\wedge T_n} = \mathbb{E}[M_{t\wedge T_n}|\mathscr{F}_s].$$

Since $|M_{t\wedge T_n}| \leq Z$, this time apply the Dominated Convergence Theorem to see that $M_{t\wedge T_n}$ converges in L^1 (to M_t) and $M_s = \mathbb{E}[M_t|\mathscr{F}_s]$.
The other two statements are immediate. $\square$

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Theorem

A continuous local martingale M with $M_0 = 0$ a.s., is a process of finite variation if and only if M is indistinguishable from zero.

Remark

Continuity is critical here.

Proof

Assume M is a continuous local martingale and of finite variation. Let

$$\tau_n = \inf\{t \geq 0 : \int_0^t |\mathrm{d}M_s| \geq n\} = \inf\{t \geq 0 : V(M)_t \geq n\},$$

which are stopping times since $V(M)_t = \int_0^t |\mathrm{d}M_s|$ is continuous and adapted.
Let $N = M^{\tau_n}$, which is bounded since

$$|N_t| = |M_{t\wedge\tau_n}| \leq |\int_0^{t\wedge\tau_n} \mathrm{d}M_u| \leq \int_0^{t\wedge\tau_n} |\mathrm{d}M_u| \leq n,$$

and hence (N_t) is a martingale.

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Let $t > 0$ and $\pi = \{0 = t_0 < t_1 < t_2 < \ldots < t_{m(\pi)} = t\}$ be a partition of $[0, t]$. Then

$$\begin{aligned}
\mathbb{E}[N_t^2] &= \sum_{i=1}^{m(\pi)} \mathbb{E}[N_{t_i}^2 - N_{t_{i-1}}^2] = \sum_{i=1}^{m(\pi)} \mathbb{E}\left[(N_{t_i} - N_{t_{i-1}})^2\right] \\
&\leq \mathbb{E}\Big[\big(\sup_{1\leq i\leq m(\pi)} |N_{t_i} - N_{t_{i-1}}|\big) \cdot \underbrace{\sum |N_{t_i} - N_{t_{i-1}}|}_{\leq V(N)_t = V(M)_{t\wedge\tau_n} \leq n}\Big] \\
&\leq n\mathbb{E}\big[\sup_{1\leq i\leq m(\pi)} |N_{t_i} - N_{t_{i-1}}|\big] \to 0 \quad \text{as } \|\pi\| \to 0
\end{aligned}$$

(where $\|\pi\|$ is the mesh of π), by the Dominated Convergence Theorem (since $|N_{t_i} - N_{t_{i-1}}| \leq V(N)_t \leq n$ and so n is a dominating function).

It then follows by Fatou's Lemma that

$$\mathbb{E}[M_t^2] = \mathbb{E}[\lim_{n\to\infty} M_{t\wedge\tau_n}^2] \leq \lim_{n\to\infty} \mathbb{E}[M_{t\wedge\tau_n}^2] = 0$$

which implies that $M_t = 0$ a.s., and so by continuity of paths, $\mathbb{P}[M_t = 0\,\forall\, t \geq 0] = 1$.