B8.2 Continuous Martingales and Stochastic Calculus

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Random processes

Constructing distributions on $(\mathbb{R}^{[0,\infty)}, \mathscr{B}(\mathbb{R}^{[0,\infty)}))$

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- Indistinguishability takes the *sample path* as the basic object of study, so we could think of $(X_t(\omega), t \geq 0)$ (the path) as a random variable taking values in the space $E^{[0,\infty)}$ (of all possible paths).
- This state space then has to be endowed with a σ-algebra of measurable sets.
- For definiteness, we take real-valued processes, so $E = \mathbb{R}$.

Definition

An *n*-dimensional *cylinder set* in $\mathbb{R}^{[0,\infty)}$ is a set of the form

$$C = \left\{\omega \in \mathbb{R}^{[0,\infty)} : \left(\omega(t_1), \ldots, \omega(t_n)\right) \in A\right\}$$

for some $0 \leq t_1 < t_2 \ldots < t_n$ and $A \in \mathscr{B}(\mathbb{R}^n)$.

- Let $\mathscr{C}$ be the family of all finite-dimensional cylinder sets and $\mathscr{B}(\mathbb{R}^{[0,\infty)})$ the σ-algebra it generates.
- This is small enough to be able to build probability measures on $\mathscr{B}(\mathbb{R}^{[0,\infty)})$ using Carathéodory's Theorem (see B8.1).
- On the other hand $\mathscr{B}(\mathbb{R}^{[0,\infty)})$ only contains events which can be defined using at most countably many coordinates.
- In particular, the set

$$\left\{\omega \in \mathbb{R}^{[0,\infty)} : \omega(t) \text{ is continuous}\right\}$$

is *not* $\mathscr{B}(\mathbb{R}^{[0,\infty)})$-measurable.

- We will have to do some work to show that many processes can be assumed to be continuous, or right continuous.
- The sample paths are then fully described by their values at times $t \in \mathbb{Q}$, which will greatly simplify the study of quantities of interest such as $\sup_{0 \le s \le t} |X_s|$ or $\tau_0(\omega) = \inf\{t \ge 0 : X_t(\omega) > 0\}$.
- A monotone class argument (see Appendix A.1) will tell us that a probability measure on $\mathscr{B}(\mathbb{R}^{[0,\infty)})$ is characterised by its finite-dimensional distributions – so if we can take continuous paths, then we only need to find the probabilities of cylinder sets to characterise the distribution of the process.

- In this section, we're going to provide a very general result about constructing continuous time stochastic processes and a criterion due to Kolmogorov which gives conditions under which there will be a version of the process with continuous paths.
- Let $\mathbb{T}$ be the set of finite increasing sequences of non-negative numbers, i.e. $\mathrm{t} \in \mathbb{T}$ if and only if $\mathrm{t} = (t_1, t_2, \ldots, t_n)$ for some n and $0 \leq t_1 < t_2 < \ldots, < t_n$.
- Suppose that for each $\mathrm{t} \in \mathbb{T}$ of length n we have a probability measure $\mathbb{P}_{\mathrm{t}}$ on $(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n))$. The collection $(\mathbb{P}_{\mathrm{t}} : \mathrm{t} \in \mathbb{T})$ is called a family of finite-dimensional (marginal) distributions.

Definition

A family of finite dimensional distributions is called *consistent* if for any $\mathrm{t} = (t_1, t_2, \ldots, t_n) \in \mathbb{T}$ and $1 \leq j \leq n$

$$\mathbb{P}_{\mathrm{t}}(A_1 \times A_2 \times \ldots \times A_{j-1} \times \mathbb{R} \times A_{j+1} \times \ldots \times A_n)$$
$$= \mathbb{P}_{\mathrm{s}}(A_1 \times A_2 \times \ldots \times A_{j-1} \times A_{j+1} \times \ldots \times A_n)$$

where $A_i \in \mathscr{B}(\mathbb{R})$ and $\mathrm{s} := (t_1, t_2, \ldots, t_{j-1}, t_{j+1}, \ldots, t_n)$.

(In other words, if we integrate out over the distribution at the jth time point then we recover the corresponding marginal for the remaining lower dimensional vector.)

If we have a probability measure $\mathbb{P}$ on $\left(\mathbb{R}^{[0,\infty)}, \mathscr{B}(\mathbb{R}^{[0,\infty)})\right)$ then it defines a consistent family of marginals via

$$\mathbb{P}_{\mathrm{t}}(A) = \mathbb{Q}(\{\omega \in \mathbb{R}^{[0,\infty)} : (\omega(t_1), \ldots, \omega(t_n)) \in A\})$$

where $\mathrm{t} = (t_1, t_2, \ldots, t_n)$, $A \in \mathscr{B}(\mathbb{R}^n)$, and we note that the set in question is in $\mathscr{B}(\mathbb{R}^{[0,\infty)})$ as it depends on finitely many coordinates.

But we'd like a converse – if I give you $\mathbb{P}_{\mathrm{t}}$, when does there exist a corresponding measure $\mathbb{P}$?

Theorem (Daniell–Kolmogorov Extension Theorem)

Let $\{\mathbb{P}_{\mathrm{t}} : \mathrm{t} \in \mathbb{T}\}$ *be a consistent family of finite-dimensional distributions. Then there exists a probability measure* $\mathbb{P}$ *on* $(\mathbb{R}^{[0,\infty)}, \mathscr{B}(\mathbb{R}^{[0,\infty)}))$ *such that for any* n*,* $\mathrm{t} = (t_1, \ldots, t_n) \in \mathbb{T}$ *and* $A \in \mathscr{B}(\mathbb{R}^n)$*,*

$$\mathbb{P}_{\mathrm{t}}(A) = \mathbb{P}[\{\omega \in \mathbb{R}^{[0,\infty)} : (\omega(t_1), \ldots, \omega(t_n)) \in A\}]. \tag{1}$$

We won't prove this (see Appendix), but notice that this defines $\mathbb{P}$ on the cylinder sets and so if we have countable additivity then the proof reduces to an application of Carathéodory's extension theorem. Uniqueness is a consequence of the Monotone Class Lemma.

This is a remarkably general result, but it doesn't allow us to say anything meaningful about the paths of the process. For that we appeal to Kolmogorov's criterion.

Theorem (Kolmogorov–Čentsov continuity criterion)

Suppose that a stochastic process $(X_t : t \leq T)$ *defined on* $(\Omega, \mathscr{F}, \mathbb{P})$ *satisfies*

$$\mathbb{E}[|X_t - X_s|^{\alpha}] \leq C|t-s|^{1+\beta}, \quad 0 \leq s,t \leq T \tag{2}$$

for some strictly positive constants α, β *and* C.
Then there exists $\tilde{X}$, *a modification of* X, *whose paths are* γ*-locally Hölder continuous* $\forall \gamma \in (0, \beta/\alpha)$ *a.s., i.e.*

$$\sup_{s,t,\in[0,T]} \frac{|\tilde{X}_t - \tilde{X}_s|}{|t-s|^{\gamma}} < \infty \quad a.s. \tag{3}$$

In particular, the sample paths of $\tilde{X}$ *are a.s. continuous (and uniformly continuous on* $[0,T]$*).*

Proof.

See appendix (not examinable) $\square$

Many more results and conditions in this direction are possible.