B8.2 Continuous Martingales and Stochastic Calculus

Strong Markov property and reflection principle

Strong Markov Property

Samuel Cohen
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UNIVERSITY OF OXFORD
Mathematical Institute

- We're going to use the sequence

$$\tau_n := \sum_{k=0}^{\infty} \frac{k+1}{2^n} 1_{\{\frac{k}{2^n} < \tau \leq \frac{k+1}{2^n}\}} + \infty 1_{\{\tau = \infty\}}$$

  of stopping times to prove an important generalisation of the Markov property for Brownian motion called the *strong* Markov property.
- Recall that the Markov property says that Brownian motion has 'no memory' – we can start it again from B_s and $B_{t+s} - B_s$ is just a Brownian motion, independent of the path followed by B up to time s.
- The strong Markov property says that the same is true if we replace s by a stopping time.

Theorem

Let $B = (B_t : t \geq 0)$ be a standard Brownian motion on the filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ and let τ be a stopping time with respect to $\{\mathscr{F}_t\}_{t\geq 0}$. Then, conditional on $\{\tau < \infty\}$, the process

$$B_t^{(\tau)} := B_{\tau+t} - B_\tau \tag{1}$$

is a standard Brownian motion independent of $\mathscr{F}_\tau$. This is called the strong Markov property of Brownian motion.

It was not until the 1940's that Doob properly formulated the strong Markov property and it was 1956 before Hunt proved it for Brownian motion.

Proof

Assume that $\tau < \infty$ a.s..
We will show that $\forall\, A \in \mathscr{F}_\tau, 0 \le t_1 < \ldots < t_p$ and continuous and bounded functions F on $\mathbb{R}^p$ we have

$$\mathbb{E}\big[1_A F(B^{(\tau)}_{t_1}, \ldots, B^{(\tau)}_{t_p})\big] = \mathbb{P}(A)\mathbb{E}\big[F(B_{t_1}, \ldots, B_{t_p})\big]. \qquad (2)$$

Granted (2), taking $A = \Omega$, we find that B and $B^{(\tau)}$ have the same finite dimensional distributions, and since $B^{(\tau)}$ has continuous paths, it must be a Brownian motion. On the other hand (as usual using a monotone class argument), (2) says that $(B^{(\tau)}_{t_1}, \ldots, B^{(\tau)}_{t_p})$ is independent of $\mathscr{F}_\tau$, and so $B^{(\tau)}$ is independent of $\mathscr{F}_\tau$.

To establish (2), first observe that by continuity of B and F,

$$F(B_{t_1}^{(\tau)},\dots,B_{t_p}^{(\tau)}) = \lim_{n\to\infty}\sum_{k=0}^{\infty} 1_{\frac{k-1}{2^n}<\tau\le\frac{k}{2^n}} F\left(B_{\frac{k}{2^n}+t_1}-B_{\frac{k}{2^n}},\dots,B_{\frac{k}{2^n}+t_p}-B_{\frac{k}{2^n}}\right)$$

and by the Dominated Convergence Theorem

$$\begin{aligned}&\mathbb{E}\left[1_A F(B_{t_1}^{(\tau)},\dots,B_{t_p}^{(\tau)})\right]\\ &= \lim_{n\to\infty}\sum_{k=0}^{\infty}\mathbb{E}\left[1_A 1_{\frac{k-1}{2^n}<\tau\le\frac{k}{2^n}} F\left(B_{\frac{k}{2^n}+t_1}-B_{\frac{k}{2^n}},\dots,B_{\frac{k}{2^n}+t_p}-B_{\frac{k}{2^n}}\right)\right].\end{aligned}$$

For $A \in \mathscr{F}_\tau$, the event $A \cap \{\frac{k-1}{2^n} < \tau \le \frac{k}{2^n}\} \in \mathscr{F}_{\frac{k}{2^n}}$, so using the simple Markov property at $k/2n$,

$$\mathbb{E}\left[1_{A\cap\{\frac{k-1}{2^n}<\tau\le\frac{k}{2^n}\}} F\left(B_{\frac{k}{2^n}+t_1} - B_{\frac{k}{2^n}}, \dots, B_{\frac{k}{2^n}+t_p} - B_{\frac{k}{2^n}}\right)\right]$$
$$= \mathbb{P}\left[A \cap \left\{\frac{k-1}{2^n} < \tau \le \frac{k}{2^n}\right\}\right] \mathbb{E}\left[F(B_{t_1}, \dots, B_{t_p})\right].$$

Sum over k to recover the desired result.
If $\mathbb{P}(\tau = \infty) > 0$, the same argument gives instead

$$\mathbb{E}\left[1_{A\cap\{\tau<\infty\}} F(B^{(\tau)}_{t_1}, \dots, B^{(\tau)}_{t_p})\right] = \mathbb{P}[A \cap \{\tau < \infty\}]\mathbb{E}\left[F(B_{t_1}, \dots, B_{t_p})\right].$$

$\square$