B8.2 Continuous Martingales and Stochastic Calculus

Brownian Motion

Lévy's construction

Samuel Cohen

Hilary Term 2021

UNIVERSITY OF OXFORD
Mathematical Institute

The following lemma establishes some classical and useful properties of normal distributions. Its proof is left as an exercise.

Lemma

(i) *Let Z, Z' be independent random variables with $Z \sim N(\mu, \Sigma)$, $Z' \sim N(\mu', \Sigma')$. Then $Z + Z' \sim N(\mu + \mu', \Sigma + \Sigma')$. Equivalently, their densities satisfy the convolution property*

$$\int_{\mathbb{R}^d} \phi_{(\mu,\Sigma)}(y)\phi_{(\mu',\Sigma')}(x-y)dy = \phi_{(\mu+\mu',\Sigma+\Sigma')}(x).$$

Lemma (Ctd...)

(ii) *If $Z_i \sim N(\mu_i, \Sigma_i)$ is a sequence of independent normal random variables such that $\mu^* = \sum_{i \in \mathbb{N}} \mu_i$ and $\Sigma^* = \sum_{i \in \mathbb{N}} \Sigma_i$ exist (i.e. the sums converge), then the sequence of partial sums $\sum_{i=1}^{n} Z_i$ converges in (L^2, and hence in) probability to a random variable with distribution*

$$\sum_{i \in \mathbb{N}} Z_i \sim N(\mu^*, \Sigma^*).$$

(iii) *If the pair (Z, Z') is a multivariate normal random variable, then Z and Z' are normal, and are independent if and only if their covariance is zero, that is, $E[(Z-\mu)(Z'-\mu')^\top] = 0$.*

We proceed as follows:

- First, we determine the value of the nth approximation X^n on the points D_n.
- Second, we use linear interpolation to define X^n_t for all values of t.
- This gives us a sequence of paths which we shall show converge.

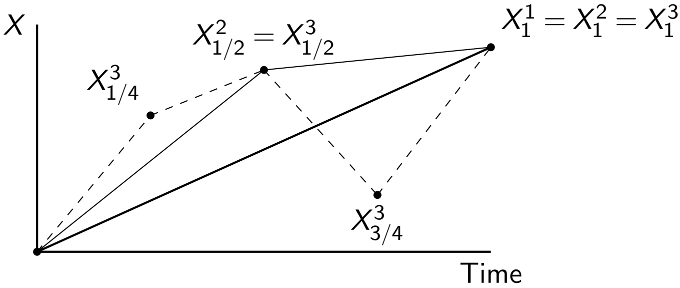

Figure: Three steps in Lévy's construction

- We begin with a countable family $\{Z_m\}$ of identically distributed random variables with $Z_m \sim N(0, I_d)$ for all m. Let $D_n = \{k2^{-n} : k, n \in \mathbb{Z}^+\}$, so that $D_n \subset D_{n+1}$, $D_0 = \mathbb{Z}^+$ and $\cup_n D_n$ is the set of Dyadic rationals.
- For simplicity of notation, let $\{Z_m\}$ be indexed by $m \in \cup_n D_n$ and $Z_0 := 0$.
- To fix the values of X_t^n for $t \in D_n$, we define

$$X_t^0 = \sum_{\{k \in D_0 : k < t\}} Z_k.$$

Next, for every $n > 0$, define $X_t^n = X_t^{n-1}$ for all $t \in D_{n-1}$. For $t \in D_n \setminus D_{n-1}$, let

$$X_t^n = X_t^{n-1} + 2^{-(n/2+1)} Z_t. \tag{1}$$

- We now linearly interpolate between these points $\{X_t^n\}_{t\in D_n}$.
- Formally, we can write the interpolation step as

$$X_t^n = X_{\lfloor t \rfloor_n} + \frac{t - \lfloor t \rfloor_n}{\lceil t \rceil_n - \lfloor t \rfloor_n}(X_{\lceil t \rceil_n} - X_{\lfloor t \rfloor_n}),$$

  where $\lfloor t \rfloor_n = \max\{s \in D_n : s \leq t\}$, $\lceil t \rceil_n = \min\{s \in D_n : s \geq t\}$.
- The use of linear interpolation is not vital to the construction, as we shall see (taking right-continuous step functions $X_t^n := X_{\lfloor t \rfloor_n}^n$ would work just as well for proving the existence of a limit, but would not immediately give continuity).
- We next show that these paths converge, in a sufficiently strong sense, to a Brownian motion.

Lemma (Lemma 3.6)

Let $\{X^n\}_{n\in\mathbb{N}}$ be a sequence of a.s. continuous functions which converge uniformly on compacts in probability to a process X, that is, for any $\varepsilon > 0$,

$$\lim_n P\left(\sup_{s\in[0,t]} \|X_s^n - X_s\| < \varepsilon\right) = 1$$

for all t. Then X is also continuous.

Remark

The uncountable supremum is measurable, as our functions are continuous (so the supremum could equally be taken over the rationals, and suprema over countable sets are always measurable).

Proof.

For fixed t, by Lemma A.7, taking a subsequence in n, we can assume that the convergence is almost sure, that is,

$$P\Big(\lim_n \big(\sup_{s\in[0,t]} \|X_s^n - X_s\|\big) = 0\Big) = 1$$

Fixing ω, this is a statement of uniform convergence of $X^{n_j} \to X$, and the continuity of the limit is classical, as for any $\varepsilon > 0$, we can find $\delta, m > 0$ such that

$$\begin{aligned}\|X_s - X_{s+\delta}\| &\leq \|X_s^{n_m} - X_s\| + \|X_{s+\delta}^{n_m} - X_{s+\delta}\| + \|X_s^{n_m} - X_{s+\delta}^{n_m}\| \\ &\leq 2 \sup_{s\in[0,t]} \{\|X_s^{n_j} - X_s\|\} + \|X_s^{n_m} - X_{s+\delta}^{n_m}\| \\ &\leq 3\varepsilon.\end{aligned}$$

$\square$

Theorem

The processes X^n defined in (1) converge a.s. uniformly on compacts to a process X. In its natural filtration, the limit is a Brownian motion starting at zero.

Proof...

Convergence. We first show that the processes converge. We consider the case where X is a Brownian motion in two dimensions, as this implies all other cases by the triangle inequality, and is notationally simpler. From our construction, we can see that

$$\sup_{s\in[0,t]} \|X_s^n - X_s^{n+1}\| = \max_{\{s\in D_{n+1}\setminus D_n : s<t\}} \|2^{-(n/2+1)} Z_s\|.$$

Proof...

The set $\{s \in D_{n+1} \setminus D_n : s < t\}$ contains at most $t2^n$ elements, and the Z_s are independent $N(0, I_d)$ random variables. It is standard that $\|Z_s\|^2$ has a χ^2-distribution with $d = 2$ degrees of freedom, so if $F(x) := P(\|Z_s\|^2 \leq x)$ is the distribution function of $\|Z_s\|^2$ we have

$$P\Big(\sup_{s\in[0,t]} \|X_s^n - X_s^{n+1}\| > \varepsilon\Big) = P\Big(\max_{\{s\in D_{n+1}\setminus D_n : s<t\}} \|Z_s\| > 2^{n/2+1}\varepsilon\Big)$$
$$\leq \sum_{\substack{\{s\in D_{n+1}\setminus D_n, \\ s<t\}}} P\big(\|Z_s\| > 2^{n/2+1}\varepsilon\big) = t2^n(1 - F(2^{n+2}\varepsilon^2)).$$

Proof...

By changing into polar coordinates, it is easy to show that $F(x) = 1 - e^{-x/2}$ (this simple form is the reason we chose $d = 2$). Therefore,

$$P\Big(\sup_{s\in[0,t]} \|X^n_s - X^{n+1}_s\| > \varepsilon\Big) \leq t2^n \exp(-2^{n/2+1}\varepsilon^2)$$

Taking N large enough that $N\log(2) - 2^{N/2+1}\varepsilon^2 < -N$, for all $n > N$ we have

$$P\Big(\sup_{s\in[0,t]} \|X^n_s - X^{n+1}_s\| > \varepsilon\Big) \leq te^{-n}.$$

Proof...

By the Borel–Cantelli Lemma, as this sequence is summable we have

$$P\Big(\sup_{s\in[0,t]} \|X^n_s - X^{n+1}_s\| > \varepsilon \text{ for infinitely many } n\Big) = 0.$$

Therefore, with probability one, the processes X^n are converging uniformly on the interval $[0,t]$. By Lemma 3.6, X is a continuous process.

X is a Brownian Motion. We now need to show that X is a Brownian motion in its natural filtration, that is, that the increment $X_t - X_s$ is normally distributed and independent of $\mathbb{F}_s = \sigma(X_u, u \leq s)$. First note that for s, t with $t \in D_n \setminus D_{n+1}$ and $\lceil s \rceil_n < t$, the random variable Z_t is not involved in the construction of X_s. Hence, as X generates the filtration and the $\{Z_u\}_{u \in \cup_n D_n}$ are independent, we see that Z_t is independent of $\mathbb{F}_s$. It is clear that if s, t are integers with $s < t$, then

$$X_t - X_s = X_t^0 - X_s^0 = \sum_{\{k \in D_0 : s < k < t\}} Z_k \sim N(0, (t-s)I_d).$$

Furthermore, in this case $X_t - X_s$ is independent of $\mathbb{F}_s$, as $Z_k = Z_{\lceil k \rceil_0}$ is independent of $\mathbb{F}_s$ for all $s < k$.

Proof...

Now suppose that the result holds for $s, t \in D_n$. Then we see that for any $u \in D_{n+1} \setminus D_n$,

$$X_u - X_{\lfloor u \rfloor_n} = \frac{X_{\lceil u \rceil_n} - X_{\lfloor u \rfloor_n}}{2} + 2^{-(n/2+1)} Z_u \sim N(0, 2^{-(n+1)} I_d)$$

which is independent of $\mathbb{F}_{\lfloor u \rfloor_n}$. Similarly,

$$X_{\lceil u \rceil_n} - X_u = -\frac{X_{\lceil u \rceil_n} - X_{\lfloor u \rfloor_n}}{2} + 2^{-(n/2+1)} Z_u \sim N(0, 2^{-(n+1)} I_d),$$

which is independent of $\mathbb{F}_{\lfloor u \rfloor_n}$. Therefore, for any $s, t \in D_{n+1}$,

$$X_t - X_s = (X_t - X_{\lfloor t \rfloor_n}) + (X_{\lfloor t \rfloor_n} - X_{\lceil s \rceil_n}) + (X_{\lceil s \rceil_n} - X_s),$$

which is the sum of three independent normal random variables, so

$$X_t - X_s \sim N(0, (t-s) I_d).$$

UNIVERSITY OF OXFORD
Mathematical Institute

The first two terms are independent of $\mathscr{F}_{\lceil s\rceil_n} \supseteq \mathscr{F}_s$. We know the last term is independent of $\mathscr{F}_{\lfloor s\rfloor_n}$, and we can compute

$$E\left[(X_{\lceil s\rceil_n} - X_s)(X_s - X_{\lfloor s\rfloor_n})^\top\right] = 0$$

so $(X_{\lceil s\rceil_n} - X_s)$ is independent of the increment $X_s - X_{\lfloor s\rfloor_n}$, as uncorrelated Gaussians are independent. As we can write

$$\mathscr{F}_s = \mathscr{F}_{\lfloor s\rfloor_n} \vee \sigma(X_s - X_{\lfloor s\rfloor_n}) \vee \sigma\big(Z_u; u \in \big]\lfloor s\rfloor_n, s\big[\big),$$

we see that $X_{\lceil s\rceil_n} - X_s$ is independent of $\mathscr{F}_s$. Therefore $X_t - X_s$ is normally distributed and independent of $\mathscr{F}_s$, as desired.

Proof...

UNIVERSITY OF OXFORD
Mathematical Institute

Finally, for any $s < t$ we can find sequences $s_n \downarrow s$, $t_n \uparrow t$ with $s_n, t_n \in D_n$ and $s_0 \le t_0$. Then $X_{t_n} - X_{s_n} \sim N(0, (t_n - s_n)I_d)$, and by continuity of X we see

$$X_t - X_s = X_{t_0} - X_{s_0} + \sum_{n=1}^{\infty} \left(X_{t_n} - X_{t_{n-1}} - X_{s_n} + X_{s_{n-1}}\right) \sim N(0, (t-s)I_d).$$

All the terms in this sum are independent of $\mathscr{F}_s$, as required. As $X_0 = 0$ by construction, we see that X is a Brownian motion starting at zero, in its natural filtration.

$\square$