UNIVERSITY OF OXFORD

Mathematical Institute

B8.2 Continuous Martingales and Stochastic Calculus

Continuous semimartingales

Functions of finite variation

Samuel Cohen

Hilary Term 2021

Oxford Mathematics

- Our original goal was to make sense of differential equations driven by ‘rough’ inputs.
- In fact, we’ll recast our differential equations as integral equations and so we must develop a theory that allows us to integrate with respect to ‘rough’ driving processes.
- The class of processes with which we work are called *semimartingales*, and we shall specialise to the continuous ones.
- We’re going to start with functions for which the integration theory that we already know is adequate – these are called functions of finite variation.
- Throughout, we assume that a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions is given.

- Throughout this section we only consider real-valued right-continuous functions on $[0, \infty)$.
- Our arguments will be shift invariant so, without loss of generality, we assume that any such function a satisfies $a(0) = 0$.

Recall the following definition.

Definition

The (total) variation of a function a over $[0, T]$ is defined as

$$V(a)_T = \sup_{\pi} \sum_{i=0}^{n_\pi - 1} |a_{t_{i+1}} - a_{t_i}|,$$

where the supremum is over partitions
$\pi = \{0 = t_0 < t_1 < \ldots < t_{n_\pi} = T\}$ of $[0, T]$.
We say that a is of *finite variation* on $[0, T]$ if $V(a)_T < \infty$. The function a is of *finite variation* if $V(a)_T < \infty$ for all $T \geq 0$ and of bounded variation if $\lim_{T\to\infty} V(a)_T < \infty$.

Remark

Note that $t \to V(a)_t$ is non-negative, right-continuous (whenever finite) and non-decreasing in t. This follows since any partition of $[0, s]$ may be included in a partition of $[0, t]$, $t \geq s$.

Proposition

The function a is of finite variation if and only if it is equal to the difference of two non-decreasing functions, a_1 and a_2.
Moreover, if a is of finite variation, then a_1 and a_2 can be chosen so that $V(a)_t = a_1(t) + a_2(t)$. If a is càdlàg then $V(a)_t$ is also càdlàg.

Proof.

$$V(a)_t - a(t) = \sup_{\pi} \sum_{i=0}^{n(\pi)-1} \Big(|a(t_{i+1}) - a(t_i)| - \big(a(t_{i+1}) - a(t_i)\big) \Big)$$

is an non-decreasing function of t, as is $V(a)_t + a(t)$. □

If we define measures μ_+, μ_- by

$$\mu_+((0,t]) = \frac{V(a)_t + a(t)}{2}, \qquad \mu_-((0,t]) = \frac{V(a)_t - a(t)}{2},$$

then we can develop a theory of integration with respect to a by declaring that

$$\int_0^t f(s)da(s) = \int_0^t f(s)\mu_+(ds) - \int_0^t f(s)\mu_-(ds),$$

provided that

$$\int_0^t |f(s)||\mu|(ds) = \int_0^t |f(s)|\left(\mu_+(ds) + \mu_-(ds)\right) < \infty.$$

We say that $\mu = \mu_+ - \mu_-$ is the *signed measure* associated with a, μ_+, μ_- is its *Jordan decomposition* and $\int_0^t f(s)da(s)$ is the *Lebesgue-Stieltjes* integral of f with respect to a.

We sometimes use the notation

$$(f \cdot a)(t) = \int_0^t f(s)da(s).$$

The function $(f \cdot a)$ will be right continuous and of finite variation whenever a is finite variation and f is a-integrable (exercise).

Example

For some $\lambda \in \mathbb{R}$, let $a(t) = 1 - e^{-\lambda t}$. Then $\mu([a,b)) = e^{-\lambda a} - e^{-\lambda b} = \int_a^b \lambda e^{-\lambda t} dt$, and we find

$$(f \cdot a)(t) = \int_0^t f(s) \lambda e^{-\lambda s} ds.$$

Similarly whenever a is any distribution function.

Remark

A function $a \in C^1$ is of finite variation and $\int f(s) da(s) = \int f(s) a'(s) ds$.