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B8.2 Continuous Martingales and Stochastic Calculus

Continuous semimartingales

Quadratic variation of a continuous local martingale

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- If our martingales are going to be interesting, then they're going to have unbounded variation.
- But remember that we said that we'd use Brownian motion as a basic building block, and that while Brownian motion has infinite variation, it has bounded *quadratic variation*, defined over $[0, T]$ by the limit (in probability)

$$\lim_{\|\pi_n\|\to 0} \sum_{j=1}^{N(\pi_n)} \left(B_{t_j} - B_{t_{j-1}}\right)^2 = T.$$

  where $\{\pi_n\}$ is a sequence of partitions.
- We are now going to see that the analogue of this process exists for any continuous local martingale. Ultimately, we shall see that the quadratic variation is in some sense a 'clock' for a local martingale, but that will be made more precise in the very last result of the course.

Theorem

Let M be a continuous local martingale. There exists a unique (up to indistinguishability) non-decreasing, continuous adapted finite variation process $(\langle M, M\rangle_t : t \geq 0)$, starting in zero, such that $(M_t^2 - \langle M, M\rangle_t : t \geq 0)$ is a continuous local martingale.
Furthermore, for any $T > 0$ and any sequence of partitions $\pi_n = \{0 = t_0^n < t_1^n < \ldots < t_{n(\pi_n)}^n = T\}$ with $\|\pi_n\| = \sup_{1 \leq i \leq n(\pi_n)}(t_i^n - t_{i-1}^n) \to 0$ as $n \to \infty$

$$\langle M, M\rangle_T = \lim_{n\to\infty} \sum_{i=1}^{n(\pi_n)} (M_{t_i^n} - M_{t_{i-1}^n})^2, \tag{1}$$

where the limit is in probability.

The process $\langle M, M\rangle$ is called the *quadratic variation of M* and is often denoted $\langle M, M\rangle_t = \langle M\rangle_t$.

Sketch of a Proof (NOT EXAMINABLE)

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Uniqueness is a direct consequence of our last theorem, since if A, A' are two such processes then

$$\left(M^2 - A - (M^2 - A')\right) = A - A'$$

is a local martingale starting in zero and of finite variation, which implies $A = A'$.

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The idea of existence is as follows. First suppose that M is *bounded*. Take a sequence of partitions $0 = t_0^n < \cdots < t_{p_n}^n = T$ with mesh tending to zero. Then check that

$$X_t^n := \sum_{i=1}^{p_n} M_{t_{i-1}^n}(M_{t_i^n \wedge t} - M_{t_{i-1}^n \wedge t})$$

is a (bounded) martingale. A direct computation gives $\lim_{n,m\to\infty} \mathbb{E}[(X_t^n - X_t^m)^2] = 0$, and by Doob's L^2-inequality

$$\lim_{n,m\to\infty} \mathbb{E}[\sup_{t\leq T}(X_t^n - X_t^m)^2] = 0.$$

By passing to a subsequence, $X^{n_k} \to Y$ almost surely on $[0, T]$ where $(Y_t)_{t\leq T}$ is a continuous process which inherits the martingale property from X.

Now observe that

$$M_{t_j^n}^2 - 2X_{t_j^n}^n = \sum_{i=1}^{j} (M_{t_i^n} - M_{t_{i-1}^n})^2 =: QV_{t_j^n}^{\pi_n}(M)$$

is non-decreasing along $t_j^n : j \leq N(\pi_n)$.

Letting $n \to \infty$, $M_t^2 - 2Y_t$ is almost surely non-decreasing and we set $\langle M, M \rangle_t = M_t^2 - 2Y_t$.

To move to a general continuous local martingale, we consider a sequence of stopped processes.

Details are in, for example, Le Gall's book. $\square$