B8.2 Continuous Martingales and Stochastic Calculus

Brownian Motion

Definition of Brownian motion

Samuel Cohen
Hilary Term 2021

UNIVERSITY OF OXFORD
Mathematical Institute

Oxford Mathematics

Our fundamental building block will be Brownian motion. It is often described as an 'infinitesimal random walk', so to motivate the definition, we take a quick look at simple (discrete time) random walk.

Definition

The discrete time stochastic process $\{S_n\}_{n\geq 0}$ is a symmetric simple random walk under the measure $\mathbb{P}$ if $S_n = \sum_{i=1}^{n} \xi_i$, where the ξ_i can take only the values ± 1, and are i.i.d. under $\mathbb{P}$ with $\mathbb{P}[\xi_i = -1] = 1/2 = \mathbb{P}[\xi_i = 1]$.

Lemma

$\{S_n\}_{n\geq 0}$ is a $\mathbb{P}$-martingale (with respect to the natural filtration) and

$$cov(S_n, S_m) = n \wedge m.$$

To obtain a 'continuous' version of simple random walk, we appeal to the Central Limit Theorem. Since $\mathbb{E}[\xi_i] = 0$ and $\mathrm{var}(\xi_i) = 1$, we have

$$\mathbb{P}\left[\frac{S_n}{\sqrt{n}} \leq x\right] \to \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \text{ as } n \to \infty.$$

More generally,

$$\mathbb{P}\left[\frac{S_{[nt]}}{\sqrt{n}} \leq x\right] \to \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi t}} e^{-y^2/2t} dy \text{ as } n \to \infty,$$

where $[nt]$ denotes the integer part of nt.

Heuristically at least, passage to the limit from simple random walk suggests the following definition of Brownian motion.

Definition (Brownian motion)

A real-valued stochastic process $\{B_t\}_{t\geq 0}$ is a $\mathbb{P}$-*Brownian motion* (or a $\mathbb{P}$-*Wiener process*) if for some real constant σ, under $\mathbb{P}$,

1. for each $s \geq 0$ and $t > 0$ the random variable $B_{t+s} - B_s$ has the normal distribution with mean zero and variance $\sigma^2 t$,
2. for each $n \geq 1$ and any times $0 \leq t_0 \leq t_1 \leq \cdots \leq t_n$, the random variables $\{B_{t_r} - B_{t_{r-1}}\}$ are independent,
3. $B_0 = 0$,
4. B_t is continuous in $t \geq 0$.

When $\sigma^2 = 1$, we say that we have a *standard* Brownian motion.

Notice in particular that for $s < t$,

$$\begin{aligned}\Gamma(s,t) = \operatorname{cov}(B_s, B_t) = \mathbb{E}[B_s B_t] &= \mathbb{E}\left[B_s^2 + B_s(B_t - B_s)\right] \\ &= \mathbb{E}[B_s^2] = s \quad (= s \wedge t = \min(s,t)).\end{aligned}$$

Using this, we can see that

$$\operatorname{var}(B_s) = s$$

and

$$\operatorname{corr}(B_s, B_t) = \sqrt{s/t}$$

We can write down the finite dimensional distributions using the independence of increments. They admit a density with respect to Lebesgue measure.

- We write $p(t,x,y)$ for the *transition density*

$$p(t,x,y) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x-y)^2}{2t}\right).$$

- This is the density (with respect to x), of B_t given $B_0 = y$. For $0 = t_0 \leq t_1 \leq t_2 \leq \ldots \leq t_n$, writing $x_0 = 0$, the joint probability density function of $B_{t_1}, \ldots, B_{t_n}$ is

$$f(x_1, \ldots, x_n) = \prod_{j=1}^{n} p(t_j - t_{j-1}, x_{j-1}, x_j).$$

We could recover the existence of Brownian motion from the general principles outlined so far (Daniell–Kolmogorov Theorem and the Kolmogorov continuity criterion), but we are next going to take a short digression to describe a beautiful (and useful) construction due to Lévy.

In fact, it's a little easier if we generalize our definition to more than one dimension, as follows:

Definition

Let μ be a probability measure on $\mathbb{R}^d$. A d-dimensional stochastic process $(B_t : t \geq 0)$ on $(\Omega, \mathscr{F}, \mathbb{P})$ is called a d-dimensional Brownian motion with initial distribution μ if

1. $\mathbb{P}[B_0 \in A] = \mu(A), \quad A \in \mathscr{B}(\mathbb{R}^d)$;
2. $\forall\, 0 \leq s \leq t$ the increment $(B_t - B_s)$ is independent of $\mathscr{F}_s = \sigma(B_u : u \leq s)$ and is normally distributed with mean 0 and covariance matrix $(t-s) \times \mathrm{I}_d$;
3. B has a.s. continuous paths.

Writing the d-dimensional Brownian motion as $B_t = (B_t^{(1)}, \ldots, B_t^{(d)})$, if $\mu(\{0\}) = 1$ then the coordinate processes $(B_t^{(i)})$, $1 \leq i \leq d$, are independent one-dimensional Brownian motions.

If $\mu(\{x\}) = 1$ for some $x \in \mathbb{R}^d$, we say that B starts at x.