B8.2 Continuous Martingales and Stochastic Calculus

UNIVERSITY OF OXFORD

Mathematical Institute

(Sub/super-)Martingales in continuous time

Convergence and regularisation theorems

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As advertised, our aim in this section is to prove that, provided the filtration satisfies 'the usual conditions', any martingale has a version with right continuous paths.

First we recall the notion of upcrossing numbers.

Definition

Let $f : I \to \mathbb{R}$ be a function defined on a subset I of $[0, \infty)$.

If $a < b$, the upcrossing number of f along $[a, b]$, which we shall denote $U([a, b], (f_t)_{t \in I})$ is the maximal integer $k \geq 1$ such that there exists a sequence $s_1 < t_1 < \cdots < s_k < t_k$ of elements of I such that $f(s_i) < a$ and $f(t_i) > b$ for every $i = 1, \ldots, k$.

If even for $k = 1$ there is no such sequence, we take $U([a, b], (f_t)_{t \in I}) = 0$. If such a sequence exists for *every* $k \geq 1$, we set $U([a, b], (f_t)_{t \in I}) = \infty$.

Upcrossing numbers are a convenient tool for studying the regularity of functions. We omit the proof of the following analytic lemma.

Lemma

Let D be a countable dense set in $[0,\infty)$ and let f be a real function defined on D. Assume that for every $T \in D$

1. *f is bounded on $D \cap [0,T]$;*
2. *for all rationals a and b such that $a < b$*

$$U([a,b],(f_t)_{t\in D\cap[0,T]}) < \infty.$$

Then the right limit $f(t+) = \lim_{s\downarrow t, s\in D} f(s)$ exists for every real $t \geq 0$, and similarly the left limit $f(t-) = \lim_{s\uparrow t, s\in D} f(s)$ exists for any real $t > 0$.

Furthermore, the function $g : \mathbb{R}_+ \to \mathbb{R}$ defined by $g(t) = f(t+)$ is càdlàg ('continue à droite avec des limites à gauche'; i.e. right continuous with left limits) at every $t > 0$.

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Lemma (Doob's upcrossing lemma in discrete time)

Let $(X_t)_{t\geq 0}$ *be a supermartingale and* F *a finite subset of* $[0,T]$*. If* $a<b$ *then*

$$\mathbb{E}\Big[U\big([a,b],(X_n : n\in F)\big)\Big] \leq \sup_{n\in F} \frac{\mathbb{E}\big[(X_n-a)^-\big]}{b-a} \leq \frac{\mathbb{E}\big[(X_T-a)^-\big]}{b-a}.$$

The last inequality follows since $(X_t-a)^-$ is a submartingale. By monotone convergence

$$\lim_{k\to\infty}\mathbb{E}\big[U\big([a,b-1/k],(X_n : n\in F)\big)\big] = \mathbb{E}\big[U\big([a,b),(X_n : n\in F)\big)\big]$$

satisfies the same bound (and similarly for other intervals)

Taking an increasing sequence F_n and setting $\cup_n F_n = F$, this immediately extends to a countable $F \subset [0, T]$. From this we deduce:

Theorem

If (X_t) is a right-continuous super-martingale and $\sup_t \mathbb{E}[X_t^-] < \infty$ then $X_\infty = \lim_{t\to\infty} X_t$ exists (convergence a.s.) and X_∞ is in L^1. In particular, a non-negative right-continuous supermartingale converges a.s. as $t \to \infty$.

Remark

Note the convergence here is almost sure, not in L^1 (that is, we usually don't have $\mathbb{E}[|X_t - X_\infty|] \to 0$ or $\mathbb{E}[X_t] \to \mathbb{E}[X_\infty]$)!

Proof

By right continuity, for any $\varepsilon > 0$ and any rational T,

$$U([a,b],(X_t)_{t\in[0,T]}) \le U([a,b-\varepsilon],(X_t)_{t\in[0,T]\cap\mathbb{Q}}).$$

By the analytic lemma, a bounded sequence $(x_n)_{n\ge 1}$ converges if and only if the number of upcrossings is finite, that is $U([a,b],(x_n)_{n\ge 1}) < \infty$ for all $a < b$ with $a, b \in \mathbb{Q}$.

By the above calculations and Doob's discrete upcrossing lemma, these statements can be taken to hold almost surely for the paths of our supermartingale X.

Hence $\{X_{t_n}\}$ converges a.s. for any sequence $t_n \uparrow \infty$, but this implies X_t converges a.s. as $t \to \infty$.

As X is a supermartingale

$$\mathbb{E}[|X_t|] = \mathbb{E}[X_t] + 2\mathbb{E}[X_t^-] \leq \mathbb{E}[X_0] + 2E[X_t^-]$$

so by Fatou's inequality

$$\mathbb{E}[|X_\infty|] = \mathbb{E}[\lim_t |X_t|] \leq \liminf_t \mathbb{E}[|X_t|] < \infty,$$

that is, $X_\infty \in L^1$. $\square$

Example

By direct calculation, we know $X_t = \exp(\theta B_t - \theta^2 t/2)$ defines a martingale, and clearly $X \geq 0$, so X_t converges almost surely as $t \to \infty$. Restricting to $t \in \mathbb{N}$, from the strong law of large numbers, we know that

$$\frac{B_t}{t} = \frac{1}{t}\sum_{s=1}^{t}(B_s - B_{s-1}) \to 0$$

and hence as $t \to \infty$

$$\theta B_t - \frac{\theta^2 t}{2} = t\left(\theta \frac{B_t}{t} - \frac{\theta^2}{2}\right) \to -\infty.$$

It follows that $X_t \to X_\infty = 0$ a.s., but

$$\mathbb{E}[|X_t - X_\infty|] = \mathbb{E}[X_t] = 1 \not\to 0 \qquad \text{and} \qquad X_t \neq \mathbb{E}[X_\infty|\mathscr{F}_t].$$