B8.2 Continuous Martingales and Stochastic Calculus

Continuous semimartingales

Processes of finite variation

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Recall that a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions is given.

Definition

An adapted right-continuous process $A = (A_t : t \geq 0)$ is called a *finite variation process* (or a *process of finite variation*) if $A_0 = 0$ and $t \mapsto A_t$ is (a function) of finite variation a.s..

Proposition

Let A be a finite variation process and K a progressively measurable process s.t.

$$\forall t \geq 0,\ \forall \omega \in \Omega,\ \int_0^t |K_s(\omega)||\mathrm{d}A_s(\omega)| < \infty.$$

Then $((K \cdot A)_t : t \geq 0)$, defined as $(K \cdot A)_t(\omega) := \int_0^t K_s(\omega)\mathrm{d}A_s(\omega)$, is a finite variation process.

Proof

The right continuity is immediate from the deterministic theory, but we need to check that $(K \cdot A)_t$ is *adapted* (and hence progressive, by Proposition **??**). For this we check that if $t > 0$ is fixed and $h : [0,t] \times \Omega \to \mathbb{R}$ is measurable with respect to $\mathscr{B}([0,t]) \otimes \mathscr{F}_t$, and if

$$\int_0^t |h(s,\omega)||dA_s(\omega)| < \infty$$

for every $\omega \in \Omega$, then

$$\int_0^t h(s,\omega) dA_s(\omega)$$

is $\mathscr{F}_t$-measurable.

Fix $t > 0$. Consider first h defined by $h(s,\omega) = 1_{(u,v]}(s)1_\Gamma(\omega)$ for $(u,v] \subseteq [0,t]$ and $\Gamma \in \mathscr{F}_t$. Then

$$(h \cdot A)_t = 1_\Gamma(A_v - A_u)$$

is $\mathscr{F}_t$-measurable. By the Monotone Class Theorem, $(h \cdot A)_t$ is $\mathscr{F}_t$-measurable for any $h = 1_G$ with $G \in \mathscr{B}([0,t]) \otimes \mathscr{F}_t$, or, more generally, any bounded $\mathscr{B}([0,t]) \otimes \mathscr{F}_t$-measurable function h.

If h is a general $\mathscr{B}([0,t])\otimes\mathscr{F}_t$-measurable function satisfying

$$\int_0^t |h(s,\omega)||\mathrm{d}A_s(\omega)| < \infty \quad \forall\omega\in\Omega,$$

then h is a pointwise limit, $h = \lim_{n\to\infty} h_n$, of simple functions with $|h| \geq |h_n|$. The integrals $\int h_n(s,\omega)dA_s(\omega)$ converge by the Dominated Convergence Theorem, and hence $\int_0^t h(s,\omega)\mathrm{d}A_s(\omega)$ is also $\mathscr{F}_t$-measurable (as a limit of $\mathscr{F}_t$-measurable functions). In particular, $(K\cdot A)_t(\omega)$ is $\mathscr{F}_t$-measurable since by progressive measurability, $(s,\omega)\mapsto K_s(\omega)$ on $[0,t]$ is $\mathscr{B}([0,t])\otimes\mathscr{F}_t$-measurable. $\square$