B8.2 Continuous Martingales and Stochastic Calculus

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(Sub/super-)Martingales in continuous time

(Super)martingale continuity

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We can now use our convergence results to give a strong continuity property for supermartingales.

Theorem

If $(X_t : t \geq 0)$ is a supermartingale then for $\mathbb{P}$-almost every $\omega \in \Omega$,

$$\forall t \in (0,\infty) \quad \lim_{r\uparrow t,\, r\in\mathbb{Q}} X_r(\omega) \text{ and } \lim_{r\downarrow t,\, r\in\mathbb{Q}} X_r(\omega) \text{ exist and are finite.} \tag{1}$$

Proof.

Fix $T > 0$. From Doob's maximal inequality and the upcrossing inequality, there exists $\Omega^T \subseteq \Omega$, with $\mathbb{P}(\Omega^T) = 1$, such that for any $\omega \in \Omega^T$

$$\forall a, b \in \mathbb{Q} \text{ with } a < b, \qquad U\big([a,b], (X_t(\omega) : t \in [0,T] \cap \mathbb{Q})\big) < \infty,$$

and

$$\sup_{t \in [0,T] \cap \mathbb{Q}} |X_t(\omega)| < \infty.$$

It follows that the limits in (1) are well defined and finite for all $t \leq T$ and $\omega \in \Omega^T$. To complete the proof, take $\Omega := \Omega^1 \cap \Omega^2 \cap \Omega^3 \cap \ldots$. $\square$

Using this, even if X is not right-continuous, its right-continuous version is a.s. well defined. The following fundamental regularisation result is again due to Doob. We begin by recalling Vitali's convergence theorem:

Theorem (Vitali convergence theorem)

Let $\{Y_k\}$ be family of random variables, and suppose $Y_k \to Y_\infty$ in probability (or a.s.). Then $Y_k \to Y_\infty$ in L^1 if and only if $\{Y_k\}$ is a uniformly integrable family.

Proof.

See appendix □

Theorem

Let X be a supermartingale with respect to a right-continuous and complete filtration $\{\mathscr{F}_t\}_{t\geq 0}$. If $t \mapsto \mathbb{E}[X_t]$ is right continuous (e.g. if X is a martingale) then X admits a modification with càdlàg paths, which is also an $\{\mathscr{F}_t\}_{t\geq 0}$-supermartingale.

Corollary

If X is a martingale then its càdlàg modification is also a martingale.

Proof

By Theorem 1, there exists $\Omega_0 \subseteq \Omega$, with $\mathbb{P}[\Omega_0] = 1$, such that the process

$$X_{t+}(\omega) = \begin{cases} \lim_{r\downarrow t,\, r\in\mathbb{Q}} X_r(\omega) & \omega \in \Omega_0 \\ 0 & \omega \notin \Omega_0 \end{cases}$$

is well defined and adapted to $\{\mathscr{F}_t\}_{t\geq 0}$. By our analytic lemma, it has càdlàg paths.

We need to check that we really have only produced a modification of X_t, that is $X_t = X_{t+}$ almost surely for all t. Let $t_n \downarrow t$ be a sequence of rationals.

Then $\{X_{t_k}\}$ is uniformly integrable (see appendix) and converges a.s. to X_{t+}. By Vitali's convergence theorem, $X_{t_k} \to X_{t+}$ in L^1, so we can pass to the limit $n \to \infty$ in the inequality $X_t \geq \mathbb{E}[X_{t_n}|\mathscr{F}_t]$ to obtain $X_t \geq \mathbb{E}[X_{t+}|\mathscr{F}_t]$.

Right continuity of $t \mapsto \mathbb{E}[X_t]$ implies $\mathbb{E}[X_{t+} - X_t] = 0$, so that $X_t = X_{t+}$ almost surely. It follows that

$$X_{s+} = X_s \geq \mathbb{E}[X_t|\mathscr{F}_s] = \mathbb{E}[X_{t+}|\mathscr{F}_s] \qquad a.s.$$

which confirms that the right-continuous modification is a supermartingale. Applying this to X and $-X$ gives the corollary.

$\square$

- Given this result, we will now often *assume* that our (sub/super)-martingales are càdlàg.
- The assumption that the filtration is right continuous is necessary. For example, let $\Omega = \{-1, +1\}$ and $\mathbb{P}[\{1\}] = \mathbb{P}[\{-1\}] = 1/2$. We set

$$X_t(\omega) = \begin{cases} 0, & 0 \le t \le 1, \\ \omega, & t > 1. \end{cases}$$

  Then X is a martingale with respect to its natural filtration (which is complete since there are no nonempty negligible sets), but no modification of X can be right continuous at $t = 1$.
- Similarly, take $X_t = f(t)$, where $f(t)$ is deterministic, non-increasing and not right continuous. Then no modification can have right continuous sample paths.