B8.2 Continuous Martingales and Stochastic Calculus

Filtrations and stopping times

Filtrations, information and adaptedness

Samuel Cohen

Hilary Term 2021

UNIVERSITY OF OXFORD
Mathematical Institute

Oxford Mathematics

- These are concepts that you already know about in the context of discrete parameter martingales.
- Our definitions here mirror what you already know, but in the continuous setting one has to be slightly more careful.
- In the end, we'll make enough assumptions to guarantee that everything goes through nicely.

Definition

We say that $\{\mathscr{F}_t\}_{t\geq 0}$ is *right continuous* if for each $t \geq 0$,

$$\mathscr{F}_t = \mathscr{F}_{t+} \equiv \cap_{\varepsilon>0} \mathscr{F}_{t+\varepsilon}.$$

We say that $\{\mathscr{F}_t\}_{t\geq 0}$ is *complete* if $(\Omega, \mathscr{F}, \mathbb{P})$ is complete (contains all subsets of the $\mathbb{P}$-null sets) and $\{A \in \mathscr{F} : \mathbb{P}[A] = 0\} \subset \mathscr{F}_0$ (and hence $\subset \mathscr{F}_t$ for all t).

Definition

A filtration $\{\mathscr{F}_t\}_{t\geq 0}$ (or the filtered space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$) is said to satisfy the *usual conditions* if it is right-continuous and complete.

- Given a filtered probability space, we can always consider a natural augmentation, replacing the filtration with $\sigma(\mathscr{F}_{t+}, \mathscr{N})$, where $\mathscr{N} = \mathscr{N}(\mathbb{P}) := \{A \in \Omega : \exists B \in \mathscr{F}$ such that $A \subseteq B$ and $\mathbb{P}[B] = 0\}$.
- The augmented filtration satisfies the usual conditions.
- In Section 6.3 we'll see that if we have a martingale with respect to a filtration that satisfies the usual conditions, then it has a right continuous version.

As in discrete time, represent 'knowing X_t at time t' by adaptedness:

Definition

A process X is *adapted* to a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ if $X(t)$ is $\mathscr{F}_t$-measurable for each $t \geq 0$ (if and only if $\mathscr{F}_t^X \subseteq \mathscr{F}_t$ for all t).

Adaptedness tells us about measurability in ω at each time t, but nothing about regularity in time. Measurability of a process tells us about regularity in time and space, but not about adaptedness. Putting these together we get the following:

Definition

A process X is $\{\mathscr{F}_t\}_{t\geq 0}$-*progressive* (or *progressively measurable*) if for each $t \geq 0$, the mapping $(s,\omega) \mapsto X_s(\omega)$ is measurable on $([0,t]\times\Omega, \mathscr{B}([0,t])\otimes\mathscr{F}_t)$.

- If X is $\{\mathscr{F}_t\}_{t\geq 0}$-progressive, then it is $\{\mathscr{F}_t\}_{t\geq 0}$-adapted, but the converse is not necessarily true.
- One can show, with difficulty, that any adapted and measurable process has a progressive modification.
- However, every right continuous $\{\mathscr{F}_t\}_{t\geq 0}$-adapted process is $\{\mathscr{F}_t\}_{t\geq 0}$-progressive and since we are interested in continuous processes, we won't need to dwell on these details.

Proposition

An adapted process (X_t) whose paths are all right-continuous (or are all left-continuous) is progressively measurable.

Proof.

We present the argument for a right-continuous X. For $t > 0$, $n \geq 1$, $k = 0, 1, 2 \ldots, 2^n - 1$ let $X_0^{(n)}(\omega) = X_0(\omega)$ and

$$X_s^{(n)}(\omega) := X_{\frac{k+1}{2^n}t}(\omega) \qquad \text{for} \qquad \frac{kt}{2^n} < s \leq \frac{k+1}{2^n}t.$$

Clearly $(X_s^{(n)} : s \leq t)$ takes finitely many values and is $\mathscr{B}([0,t]) \otimes \mathscr{F}_t$–measurable. By right continuity, $X_s(\omega) = \lim_{n\to\infty} X_s^{(n)}(\omega)$, and hence is also measurable (as a limit of measurable mappings). $\square$

Usually we shall consider the natural filtration associated with a process and don't specify it explicitly. On the other hand, sometimes we suppose that we are *given* a filtration $\mathscr{F}_t$ In this case,

Definition

A process B is an $\{\mathscr{F}_t\}_{t\geq 0}$-Brownian motion if it is adapted to $\{\mathscr{F}_t\}_{t\geq 0}$, $B_0 = 0$, B has continuous paths, $B_t - B_s \sim N(0, t-s)$ for $t > s$ and $B_t - B_s$ is independent of $\mathscr{F}_s$ for all $t > s$. Equivalently, B is adapted, a Brownian motion in its own filtration, and $B_t - B_s$ is independent of $\mathscr{F}_s$ for all $t > s$.

Example

Let B be a Brownian motion (in its natural filtration), and let $\mathscr{F}_t = \sigma(B_s; s \leq t) \vee \sigma(B_T)$ for some $T > 0$. Then B is *not* an $\{\mathscr{F}_t\}_{t\geq 0}$-Brownian motion.