B8.2 Continuous Martingales and Stochastic Calculus

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Itô's formula and its applications

Applications of Itô's formula

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- As a first application, suppose that M is a continuous local martingale and A is a process of finite variation.
- Then $\langle M, A\rangle \equiv 0$ and applying Itô's formula with $X^1 = M$ and $X^2 = A$ yields

$$
\begin{aligned}
F(M_t, A_t) = F(M_0, A_0) &+ \int_0^t \frac{\partial F}{\partial m}(M_s, A_s)\mathrm{d}M_s \\
&+ \int_0^t \frac{\partial F}{\partial a}(M_s, A_s)\mathrm{d}A_s + \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial m^2}(M_s, A_s)\mathrm{d}\langle M\rangle_s.
\end{aligned}
$$

Note that this gives us the semimartingale decomposition of $F(M_t, A_t)$ and we can, for example, read off the conditions on F under which we recover a local martingale. (Even the fact that $F(M_t, A_t)$ is a semimartingale isn't obvious otherwise!)

In particular, taking $F(x,y) = \exp(\lambda x - \frac{\lambda^2}{2} y)$ with $X^1 = M$ and $X^2 = \langle M, M \rangle$, we obtain:

Proposition

Let M be a continuous local martingale and $\lambda \in \mathbb{R}$. Then

$$\mathscr{E}^\lambda(M)_t := \exp\left(\lambda M_t - \frac{\lambda^2}{2}\langle M\rangle_t\right), \quad t \geq 0, \tag{1}$$

is a continuous local martingale. In fact the same holds true for any $\lambda \in \mathbb{C}$ with the real and imaginary parts being local martingales.

Proof.

Let $F(x,y) = \exp\left(\lambda x - \frac{\lambda^2}{2} y\right)$. $F \in C^2(\mathbb{R}^2, \mathbb{C})$ so we may apply Itô's formula to $\mathscr{E}^\lambda(M)_t = F(M_t, \langle M \rangle_t)$. Computing the partial derivatives and simplifying gives:

$$\mathscr{E}^\lambda(M)_t = \mathscr{E}^\lambda(M)_0 + \int_0^t \frac{\partial}{\partial x} F^\lambda(M_s, \langle M \rangle_s) \mathrm{d}M_s.$$

$\square$

Note that we have $\frac{\partial}{\partial x} F(x,y) = \lambda F(x,y)$ so we could have written this as

$$\mathscr{E}^\lambda(M)_t = \mathscr{E}^\lambda(M)_0 + \lambda \int_0^t \mathscr{E}^\lambda(M)_s \mathrm{d}M_s$$

or in 'differential form' as

$$\mathrm{d}\mathscr{E}^\lambda(M)_t = \lambda \mathscr{E}^\lambda(M)_t \mathrm{d}M_t$$

which shows $\mathscr{E}^\lambda(M)$ solves the stochastic exponential differential equation driven by M: $\mathrm{d}Y_t = \lambda Y_t \mathrm{d}M_t$.

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Lévy's characterization

Here is a beautiful application of exponential martingales:

Theorem (Lévy's characterisation of Brownian motion)

Let M be a continuous local martingale starting at zero. Then M is a standard Brownian motion if and only if $\langle M \rangle_t = t$ a.s. for all $t \geq 0$.

Proof

The 'only if' direction is easy; we consider the 'if' direction. Suppose M is a continuous local martingale starting in zero with $\langle M \rangle_t = t$ a.s. for all $t \geq 0$. Then, by Proposition 1,

$$\exp\left(\mathrm{i}\xi M_t + \frac{\xi^2}{2} t\right), \quad t \geq 0$$

is a local martingale for any $\xi \in \mathbb{R}$ and, since it is bounded, it is a martingale. Let $0 \leq s < t$. We have

$$\mathbb{E}\left[\exp\left(\mathrm{i}\xi M_t + \frac{\xi^2}{2} t\right) \middle| \mathscr{F}_s\right] = \exp\left(\mathrm{i}\xi M_s + \frac{\xi^2}{2} s\right)$$

which we can rewrite as

$$\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\xi(M_t - M_s)} \middle| \mathscr{F}_s\right] = \mathrm{e}^{-\frac{\xi^2}{2}(t-s)}. \tag{2}$$

In other words, $M_t - M_s$ is centred Gaussian with (conditional) variance $t - s$.

It follows also from (2) that for $A \in \mathscr{F}_s$,

$$\mathbb{E}\left[1_A e^{i\xi(M_t - M_s)}\right] = \mathbb{P}[A]\mathbb{E}\left[e^{i\xi(M_t - M_s)}\right],$$

so fixing $A \in \mathscr{F}_s$ with $\mathbb{P}[A] > 0$ and writing $\mathbb{P}_A = \mathbb{P}[\cdot \cap A]/\mathbb{P}[A]$ (which is a probability measure on $\mathscr{F}_s$) for the conditional probability given A, we have that $M_t - M_s$ has the same distribution under $\mathbb{P}$ as under $\mathbb{P}_A$.

Therefore, $M_t - M_s$ is independent of $\mathscr{F}_s$ and we have that M is an $\{\mathscr{F}_t\}_{t \geq 0}$-Brownian motion. $\square$

- So the quadratic variation is capturing all the information about M.
- This is surprising – recall that it is a special property of Gaussians that they are characterised by their means and the variance-covariance matrix, but in general we need to know much more.
- It also shows we didn't really need the Gaussian assumption in our definition of Brownian motion, it's guaranteed by the independence and variance assumptions.

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Dambis–Dubins–Schwarz Theorem

It turns out that what we just saw for Brownian motion has a powerful consequence for all continuous local martingales

- they are characterised by their quadratic variation and,
- in fact, they are all time changes of Brownian motion.

Theorem (Dambis–Dubins–Schwarz Theorem)

Let M be an $(\{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P}))$-continuous local martingale with $M_0 = 0$ and $\langle M\rangle_\infty = \infty$ a.s. Let $\tau_s := \inf\{t \geq 0 : \langle M\rangle_t > s\}$. Then the process B defined by $B_s := M_{\tau_s}$, is an $(\{\mathscr{F}_{\tau_s}\}_{s\geq 0}, \mathbb{P})$-Brownian motion and $M_t = B_{\langle M\rangle_t}$, $\forall t \geq 0$ a.s.

Proof

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Note that τ_s is the first hitting time of an open set (s, ∞) for an adapted process $\langle M \rangle$ with continuous sample paths, and hence τ_s is a stopping time (recall that $\{\mathscr{F}_t\}_{t \geq 0}$ is right-continuous).

Further, $\langle M \rangle_\infty = \infty$ a.s. implies that $\tau_s < \infty$ a.s. The process $(\tau_s : s \geq 0)$ is non-decreasing and right-continuous (in fact $s \to \tau_s$ is the right-continuous inverse of $t \to \langle M \rangle_t$).

Let $\mathscr{G}_s := \mathscr{F}_{\tau_s}$. Note that it satisfies the usual conditions.

The process B is right continuous by continuity of M and right-continuity of τ. We have

$$\lim_{u \uparrow s} B_u = \lim_{u \uparrow s} M_{\tau_u} = M_{\tau_{s-}}.$$

But$[\tau_{s-}, \tau_s]$ is either a point or an interval of constancy of $\langle M \rangle$.

The latter are known (exercise) to coincide a.s. with the intervals of constancy of M and hence $M_{\tau_{s-}} = M_{\tau_s} = B_s$ so that B has a.s. continuous paths.

To conclude that B is a $(\mathscr{G}_s)$-Brownian motion, by Lévy's theorem, it remains to show that (B_s) and $(B_s^2 - s)$ are $(\mathscr{G}_s)$-local martingales.

Note that M^{τ_n} and $(M^{\tau_n})^2 - \langle M \rangle^{\tau_n}$ are uniformly integrable martingales. Taking $0 \le u < s < n$ and applying the Optional Stopping Theorem we obtain

$$\mathbb{E}[B_s|\mathscr{G}_u] = \mathbb{E}[M^{\tau_n}_{\tau_s}|\mathscr{F}_{\tau_u}] = M^{\tau_n}_{\tau_u} = M_{\tau_u} = B_u$$

and

$$\begin{aligned}\mathbb{E}[B_s^2 - s|\mathscr{G}_u] &= \mathbb{E}\left[(M^{\tau_n}_{\tau_s})^2 - \langle M \rangle^{\tau_n}_{\tau_s}|\mathscr{F}_{\tau_u}\right] \\ &= (M^{\tau_n}_{\tau_u})^2 - \langle M \rangle^{\tau_n}_{\tau_u} = (M_{\tau_u})^2 - \langle M \rangle_{\tau_u} \\ &= B_u^2 - u,\end{aligned}$$

where we used continuity of $\langle M \rangle$ to write $\langle M \rangle_{\tau_u} = u$. It follows that B is indeed a $(\mathscr{G}_s)$-Brownian motion.

Finally, $B_{\langle M \rangle_t} = M_{\tau_{\langle M \rangle_t}} = M_t$, again since the intervals of constancy of M and of $\langle M \rangle$ coincide a.s. so that $s \to \tau_s$ is constant on $[t, \tau_{\langle M \rangle_t}]$. $\square$