UNIVERSITY OF OXFORD

Mathematical Institute

B8.2 Continuous Martingales and Stochastic Calculus

Stochastic Integration

Stochastic dominated convergence

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Proposition (Stochastic Dominated Convergence Theorem)

Let X be a continuous semimartingale and K^n a sequence in $L(X)$ with $K^n_t \to 0$ as $n \to \infty$ a.s. for all t. Further suppose that $|K^n_t| \leq K_t$ for all n where $K \in L(X)$. Then $K^n \bullet X$ converges to zero in probability and, more precisely,

$$\forall t \geq 0 \quad \sup_{s \leq t} \left| \int_0^s K^n_u \mathrm{d}X_u \right| \longrightarrow 0 \textit{ in probability as } n \to \infty.$$

Proof

We can treat the finite variation part, $X_0 + A$, and the local martingale part, M, separately. For the first, note that

$$\begin{aligned}\left|\int_0^t K_u^n \mathrm{d}A_u\right| &= \left|\int_0^t K_u^n \mathrm{d}A_u^+ - \int_0^t K_u^n \mathrm{d}A_u^-\right| \\ &\leq \int_0^t |K_u^n| \mathrm{d}A_u^+ + \int_0^t |K_u^n| \mathrm{d}A_u^- = \int_0^t |K_u^n||\mathrm{d}A_u|.\end{aligned}$$

The a.s. pointwise convergence of K^n to 0, together with the bound $|K^n| \leq K$, allow us to apply the (usual) Dominated Convergence Theorem to conclude that, for any $t > 0$, $\int_0^t |K_u^n||\mathrm{d}A_u|$ converges to 0 a.s. (in fact, as $\int_0^t |K_u^n||\mathrm{d}A_u|$ is non-decreasing in t, the convergence is uniform on any compact interval).

For the continuous local martingale part M, let (τ_m) be a reducing sequence such that $M^{\tau_m} \in \mathscr{H}_0^{2,c}$ and $K \in \mathrm{L}^2(M^{\tau_m})$. Then, by the Itô isometry,

$$\begin{aligned}\|K^n \bullet M^{\tau_m}\|^2_{\mathscr{H}^{2,c}} &= \mathbb{E}\left[\left(\int_0^{\tau_m} K_t^n \mathrm{d}M_t\right)^2\right] \\ &= \mathbb{E}\left[\int_0^{\infty} (K_t^n)^2 1_{[0,\tau_m]}(t)\mathrm{d}\langle M\rangle_t\right] = \|K^n\|^2_{\mathrm{L}^2(M^{\tau_m})}.\end{aligned}$$

The right hand side tends to zero by the usual Dominated Convergence Theorem. For a fixed $t \geq 0$, and any given $\varepsilon > 0$, we may take m large enough that $\mathbb{P}[\tau_m \leq t] \leq \varepsilon/2$.

We then have

$$\mathbb{P}\left[\sup_{s\le t}|(K^n\bullet M)_s|>\varepsilon\right]\le\mathbb{P}\left[\sup_{s\le t\wedge\tau_m}|(K^n\bullet M)_s|>\varepsilon\right]+\varepsilon/2$$
$$\le\frac{1}{\varepsilon^2}\|K^n\bullet M^{\tau_m}\|^2_{\mathscr{H}^{2,c}}+\varepsilon/2\le\varepsilon,$$

for n large enough. □

From this we can also confirm that even in their most general form our stochastic integrals can be thought of as limits of integrals of simple functions.

Proposition

Let X be a continuous semimartingale and K a left-continuous process in $L(X)$. If π^n is a sequence of partitions of $[0,t]$ with mesh converging to zero then

$$\sum_{t_i\in\pi^n} K_{t_i}(X_{t_{i+1}} - X_{t_i}) \longrightarrow \int_0^t K_s \mathrm{d}X_s \text{ in probability as } n\to\infty.$$