B8.2 Continuous Martingales and Stochastic Calculus

(Sub/super-)Martingales in continuous time

Doob's maximal inequalities

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- Doob was the person who placed martingales on a firm mathematical foundation (beginning in the 1940's).
- He initially called them 'processes with the property E', but reverted to the term martingale in his monumental book.

- Doob's inequalities are fundamental to proving convergence theorems for martingales.
- You already encountered them in the discrete setting and we shall recall those results that underpin our proofs in the continuous world here.
- They allow us to control the running maximum of a martingale.

The following results are standard in discrete time (see Appendix)

Theorem

If $(X_n)_{n\geq 0}$ is a discrete martingale (or a nonnegative submartingale) w.r.t. some filtration $(\mathscr{F}_n)$, then for any $N \in \mathbb{N}$, $p \geq 1$ and $\lambda > 0$,

$$\lambda^p \mathbb{P}\left[\sup_{n\leq N} |X_n| \geq \lambda\right] \leq \mathbb{E}\left[|X_N|^p\right]$$

and for any $p > 1$

$$E\left[|X_N|^p\right] \leq E\left[\sup_{n\leq N} |X_n|^p\right] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}\left[|X_N|^p\right].$$

We'd now like to extend this to continuous time.

Basic strategy:

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- Suppose that X is indexed by $t \in [0, \infty)$.
- Take a countable dense set D in $[0, T]$, e.g. $D = \mathbb{Q} \cap [0, T]$, and an increasing sequence of finite subsets $D_n \subseteq D_{n+1} \subseteq D$ such that $\cup_{n=1}^{\infty} D_n = D$.
- The above inequalities hold for X indexed by $t \in D_n \cup \{T\}$. Monotone convergence then yields the result for $t \in D$. If X has regular sample paths (e.g. right continuous) then the supremum over a countable dense set in $[0, T]$ is the same as over the *whole* of $[0, T]$.

Theorem (Doob's maximal and L^p inequalities)

If $(X_t)_{t\geq 0}$ is a right continuous martingale or nonnegative sub-martingale, then for any $T \geq 0$, $\lambda > 0$,

$$\begin{aligned} \mathbb{P}\Big[\sup_{t\leq T}|X_t| \geq \lambda\Big] &\leq \frac{1}{\lambda^p}\mathbb{E}\big[|X_T|^p\big], \quad p \geq 1 \\ E\Big[\sup_{t\leq T}|X_t|^p\Big] &\leq \Big(\frac{p}{p-1}\Big)^p\mathbb{E}\big[|X_T|^p\big], \quad p > 1. \end{aligned} \tag{1}$$

As an application of Doob's maximal inequality, we derive a useful bound for Brownian motion.

Proposition

Let $(B_t)_{t\geq 0}$ be Brownian motion and $S_t = \sup_{u\leq t} B_u$. For any $\lambda > 0$ we have

$$\mathbb{P}[S_t \geq \lambda t] \leq e^{-\frac{\lambda^2 t}{2}}.$$

Proof.

Recall that $e^{\alpha B_t - \alpha^2 t/2}$, $t \geq 0$, is a non-negative martingale. It follows that, for $\alpha \geq 0$,

$$
\begin{aligned}
\mathbb{P}[S_t \geq \lambda t] &\leq \mathbb{P}\left[\sup_{u \leq t}\left(e^{\alpha B_u - \alpha^2 t/2}\right) \geq e^{\alpha\lambda t - \alpha^2 t/2}\right] \\
&\leq \mathbb{P}\left[\sup_{u \leq t}\left(e^{\alpha B_u - \alpha^2 u/2}\right) \geq e^{\alpha\lambda t - \alpha^2 t/2}\right] \\
&\leq e^{-\alpha\lambda t + \alpha^2 t/2} \underbrace{\mathbb{E}\left[e^{\alpha B_t - \alpha^2 t/2}\right]}_{=1}.
\end{aligned}
$$

The bound now follows since $\min_{\alpha \geq 0} e^{-\alpha\lambda t + \alpha^2 t/2} = e^{-\lambda^2 t/2}$ (with the minimum achieved when $\alpha = \lambda$). $\square$

- In the next subsection, we are going to show that even if a supermartingale is not right continuous, it has a right continuous version (this is Doob's Regularisation Theorem).
- To prove this, we need a slight variant of the maximal inequality – this time for a supermartingale – which in turn relies on Doob's Optional Stopping (or Sampling) Theorem for discrete supermartingales.

Theorem (Doob's Optional Stopping Theorem for discrete supermartinagles (bounded time case))

If $(Y_n)_{n\geq 1}$ is a supermartingale, then for any choice of bounded stopping times S and T such that $S \leq T$, we have

$$Y_S \geq \mathbb{E}[Y_T | \mathscr{F}_S].$$

Here's the version of the maximal inequality that we shall need.

Proposition

Let $(X_t : t \geq 0)$ *be a supermartingale. Then*

$$\mathbb{P}\left[\sup_{t\in[0,T]\cap\mathbb{Q}} |X_t| \geq \lambda\right] \leq \frac{1}{\lambda}\left(2\mathbb{E}[|X_T|] + \mathbb{E}[|X_0|]\right), \quad \forall \lambda, T > 0. \quad (2)$$

In particular, $\sup_{t\in[0,T]\cap\mathbb{Q}} |X_t| < \infty$ *a.s.*

Proof

Take a sequence of rational numbers $0 = t_0 < t_1 < \ldots < t_n = T$. Applying the optional stopping theorem with $S = \min\{t_i : X_{t_i} \geq \lambda\} \wedge T$, we obtain

$$\mathbb{E}[X_0] \geq \mathbb{E}[X_S] \geq \lambda \mathbb{P}[\sup_{1\leq i\leq n} X_{t_i} \geq \lambda] + \mathbb{E}[X_T 1_{\sup_{1\leq i\leq n} X_{t_i} < \lambda}].$$

Rearranging,

$$\lambda \mathbb{P}(\sup_{1\leq i\leq n} X_{t_i} \geq \lambda) \leq \mathbb{E}[X_0] + \mathbb{E}[X_T^-]$$

(where $X_T^- = -\min(X_T, 0)$).

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Now X_T^- is a non-negative submartingale and so we can apply Doob's inequality directly to it, from which

$$\lambda \mathbb{P}(\sup_{1 \le i \le n} X_{t_i}^- \ge \lambda) \le \mathbb{E}[X_T^-],$$

and, since $\mathbb{E}[X_T^-] \le \mathbb{E}[|X_T|]$, taking the (monotone) limit in nested sequences in $[0, T] \cap \mathbb{Q}$, gives the result. $\square$