B8.2 Continuous Martingales and Stochastic Calculus

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Continuous semimartingales

Finite variation integrals

Samuel Cohen
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Oxford Mathematics

Our integral can also be obtained in the usual way:

- Let $f : [0, T] \to \mathbb{R}$ be left-continuous and $0 = t_o^n < t_1^n < \cdots < t_{p_n}^n = T$ be a sequence of partitions of $[0, T]$ with mesh tending to zero. Then

$$\int_0^T f(s)da(s) = \lim_{n\to\infty} \sum_{i=1}^{p_n} f(t_{i-1}^n)\big(a(t_i^n) - a(t_{i-1}^n)\big).$$

- The proof is easy: let $f_n : [0, T] \to \mathbb{R}$ be defined by $f_n(s) = f(t_{i-1}^n)$ if $s \in (t_{i-1}^n, t_i^n]$, $1 \le i \le p_n$, and $f_n(0) = 0$. Then

$$\sum_{i=1}^{p_n} f(t_{i-1}^n)\big(a(t_i^n) - a(t_{i-1}^n)\big) = \int_{[0,T]} f_n(s)\mu(ds),$$

where μ is the signed measure associated with a. The desired result now follows by the Dominated Convergence Theorem.

- In the argument above, f_n took the value of f at the *left* endpoint of each interval.
- In the finite variation case, we could equally have approximated by f_n taking the value of f at the midpoint of the interval, or the right hand endpoint, or any other point in between, but the limits could differ if a were not continuous.

We now will state some properties of the finite variation integral (and prove some of them).

Proposition (Associativity)

Let a be of finite variation as above and f, g measurable functions, f is a-integrable and g is $(f \cdot a)$-integrable. Then gf is a-integrable and

$$\int_0^t g(s)\mathrm{d}(f \cdot a)(s) = \int_0^t g(s)f(s)\mathrm{d}a(s).$$

In our 'dot'-notation:

$$g \cdot (f \cdot a) = (gf) \cdot a. \tag{1}$$

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Proposition (Stopping)

Let a be of finite variation as above and fix $t \geq 0$. Set $a^t(s) = a(t \wedge s)$. Then a^t is of finite variation and for any measurable a-integrable function f

$$\int_0^{u \wedge t} f(s)\mathrm{d}a(s) = \int_0^u f(s)\mathrm{d}a^t(s) = \int_0^u f(s)1_{[0,t]}(s)\mathrm{d}a(s), \quad u \in [0, \infty].$$

Proposition (Integration by parts)

Let a and b be two right-continuous functions of finite variation with $a(0) = b(0) = 0$. Then for any t

$$a(t)b(t) = \int_0^t a(s-)\mathrm{d}b(s) + \int_0^t b(s-)\mathrm{d}a(s) + \sum_{s\in[0,t]} \Delta a(s)\Delta b(s)$$

where $\Delta a(t) = a(t) - a(t-)$ and $a(t-) = \lim_{s\uparrow t} a(s)$.

Remark

As a and b are right-continuous they have at most countably many discontinuities, and as they are of finite variation, the left-limits exist.

Sketch.

For a partition π_n, take a telescoping sum

$$
\begin{aligned}
a(t)b(t) &= \sum_{t_i \in \pi_n} \big(a(t_i)b(t_i) - a(t_{i-1})b(t_{i-1})\big) \\
&= \sum_{t_i \in \pi_n} a(t_{i-1})\big(b(t_i) - b(t_{i-1})\big) + \sum_{t_i \in \pi_n} b(t_{i-1})\big(a(t_i) - a(t_{i-1})\big) \\
&\quad + \sum_{t_i \in \pi_n} \big(a(t_i) - a(t_{i-1})\big)\big(b(t_i) - b(t_{i-1})\big).
\end{aligned}
$$

By dominated convergence, these converge to the stated integrals. $\square$

Proposition (Chain-rule)

If F is a C^1 function and a is continuous of finite variation, then $F(a(t))$ is also of finite variation and

$$F(a(t)) = F(a(0)) + \int_0^t F'(a(s))\mathrm{d}a(s).$$

Proof.

The statement is trivially true for $F(x) = x$. Now by integration by parts, it is straightforward to check that if the statement is true for F, then it is also true for $xF(x)$. Hence, by induction, the statement holds for all polynomials. To complete the proof, approximate $F \in C^1$ by a sequence of polynomials. $\square$

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Proposition (Change of variables)

If a is non-decreasing and right-continuous then so is its 'right inverse'

$$c(s) := \inf\{t \geq 0 : a(t) > s\},$$

where $\inf \emptyset = +\infty$. *Let* $a(0) = 0$. *Then, for any Borel measurable function* $f \geq 0$ *on* $\mathbb{R}_+$, *we have*

$$\int_0^\infty f(u)\mathrm{d}a(u) = \int_0^{a(\infty)} f(c(s))\mathrm{d}s.$$

Proof.

If $f(u) = 1_{[0,v]}(u)$, then the claim becomes

$$a(v) = \int_0^\infty 1_{\{c(s) \leq v\}} ds = \inf\{s : c(s) > v\},$$

and equality holds by definition of c. Take differences to get indicators of sets $(u, v]$. The Monotone Class Theorem allows us to extend to functions of compact support and then take increasing limits to obtain the formula in general. $\square$