B8.2 Continuous Martingales and Stochastic Calculus

Continuous semimartingales

$\mathscr{H}^2$ space

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Our theory of integration is going to be an 'L^2-theory'. We first need to introduce the martingales with which we are going to work.

- We are going to think of them as being defined up to indistinguishability – nothing changes if we change the process on a null set.
- Think of this as analogous to considering Lebesgue integrable functions as being defined 'almost everywhere'.

Definition

Let $\mathscr{H}^2$ be the space of L^2-bounded càdlàg martingales, i.e.

$$(\{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})\text{–martingales } M \text{ s.t. } \sup_{t\geq 0} \mathbb{E}[M_t^2] < \infty,$$

and $\mathscr{H}^{2,c}$ the subspace consisting of continuous L^2-bounded martingales. Finally, let $\mathscr{H}_0^{2,c} = \{M \in \mathscr{H}^{2,c} : M_0 = 0 \text{ a.s.}\}$.

We note that the space $\mathscr{H}^2$ is also sometimes denoted $\mathscr{M}^2$.

It follows from Doob's L^2-inequality that

$$\mathbb{E}\left[\sup_{t\geq 0} M_t^2\right] \leq 4\sup_{t\geq 0}\mathbb{E}[M_t^2] < +\infty, \quad M \in \mathscr{H}^2.$$

Consequently, $\{M_t : t \geq 0\}$ is bounded by a square integrable random variable ($\sup_{t\geq 0}|M_t|$) and in particular is uniformly integrable. It follows from the martingale convergence theorem that $M_t = \mathbb{E}[M_\infty|\mathscr{F}_t]$ for some square integrable random variable M_∞.

Conversely, we can start with a random variable $Y \in \mathrm{L}^2(\Omega, \mathscr{F}_\infty, \mathbb{P})$ and define a martingale $M_t := \mathbb{E}[Y|\mathscr{F}_t] \in \mathscr{H}^2$ (and $M_\infty = Y$).

Two L^2-bounded martingales M, M' are indistinguishable if and only if $M_\infty = M'_\infty$ a.s. and so if we endow $\mathscr{H}^2$ with the norm

$$\|M\|_{\mathscr{H}^2} := \sqrt{\mathbb{E}[M_\infty^2]} = \|M_\infty\|_{\mathrm{L}^2(\Omega,\mathscr{F}_\infty,\mathbb{P})}, \quad M \in \mathscr{H}^2, \tag{1}$$

then $\mathscr{H}^2$ can be identified with the familiar $\mathrm{L}^2(\Omega, \mathscr{F}_\infty, \mathbb{P})$ space.

Remark

It's worth noticing that $\|M\|^2_{\mathscr{H}^2} = \mathbb{E}[M_\infty^2] = \mathrm{var}(M_\infty)$ *(recalling that* $\mathbb{E}[M_\infty] = 0$*).*

Theorem

$\mathscr{H}^{2,c}$ is a closed subspace of $\mathscr{H}^2$.

Proof.

This is almost a matter of writing down definitions. Suppose that the sequence $M^n \in \mathscr{H}^{2,c}$ converges in $\|\cdot\|_{\mathscr{H}^2}$ to some $M \in \mathscr{H}^2$. By Doob's L^2-inequality

$$\mathbb{E}\left[\sup_{t\geq 0}|M_t^n - M_t|^2\right] \leq 4\|M^n - M\|^2_{\mathscr{H}^2} \longrightarrow 0, \quad \text{as } n \to \infty.$$

Passing to a subsequence, we have $\sup_{t\geq 0}|M_t^{n_k} - M_t| \to 0$ a.s. and hence M has continuous paths a.s.,which completes the proof. $\square$

For continuous local martingales, our description of $\mathscr{H}^2$, in particular the norm in (1) can be re-expressed in terms of the quadratic variation:

Theorem

Let M be a continuous local martingale with $M_0 \in L^2$.

1. *TFAE*
   1.1 *M is a martingale, bounded in L^2;*
   1.2 $\mathbb{E}[\langle M, M\rangle_\infty] < \infty$.

   Furthermore, if these properties hold, $M_t^2 - \langle M, M\rangle_t$ is a uniformly integrable martingale and, in particular, $\mathbb{E}[M_\infty^2] = \mathbb{E}[M_0^2] + \mathbb{E}[\langle M, M\rangle_\infty]$.
2. *TFAE*
   2.1 *M is a martingale and $M_t \in L^2$ for every $t \geq 0$;*
   2.2 $\mathbb{E}[\langle M, M\rangle_t] < \infty$ *for every $t \geq 0$.*

   Furthermore, if these properties hold, $M_t^2 - \langle M, M\rangle_t$ is a martingale.

Proof

The second statement will follow from the first on applying it to $M_{t\wedge a}$ for every choice of $a \geq 0$.
To prove the first set of equivalences, without loss of generality, suppose that $M_0 = 0$ (or replace M by $M - M_0$).

Suppose that M is a martingale, bounded in L^2. Doob's L^2-inequality implies that for every $T > 0$,

$$\mathbb{E}[\sup_{0\leq t\leq T} M_t^2] \leq 4\mathbb{E}[M_T^2],$$

and so, letting $T \to \infty$,

$$\mathbb{E}[\sup_{t\geq 0} M_t^2] \leq 4 \sup_{t\geq 0} \mathbb{E}[M_t^2] = C < \infty.$$

Let $S_n = \inf\{t \geq 0 : \langle M, M\rangle_t \geq n\}$. Then the continuous local martingale $M^2_{t\wedge S_n} - \langle M, M\rangle_{t\wedge S_n}$ is dominated by $\sup_{s\geq 0} M^2_s + n$, which is integrable.

By Proposition 7.17 (lecture before now) this continuous local martingale is a uniformly integrable martingale, so $\mathbb{E}[M^2_{t\wedge S_n} - \langle M\rangle_{t\wedge S_n}] = 0$, and hence

$$\mathbb{E}[\langle M, M\rangle_{t\wedge S_n}] = \mathbb{E}[M^2_{t\wedge S_n}] \leq \mathbb{E}[\sup_{s\geq 0} M^2_s] \leq C < \infty.$$

Let n and then t tend to infinity and use the Monotone Convergence Theorem to obtain $\mathbb{E}[\langle M, M\rangle_\infty] < \infty$.

Conversely, assume that $\mathbb{E}[\langle M, M\rangle_\infty] < \infty$.

Set $T_n = \inf\{t \geq 0 : |M_t| \geq n\}$. Then the continuous local martingale $M^2_{t\wedge T_n} - \langle M, M\rangle_{t\wedge T_n}$ is dominated by $n^2 + \langle M, M\rangle_\infty$ which is integrable.

From Proposition 7.17 again, this continuous local martingale is a uniformly integrable martingale and hence for every $t \geq 0$,

$$\mathbb{E}[M^2_{t\wedge T_n}] = \mathbb{E}[\langle M, M\rangle_{t\wedge T_n}] \leq \mathbb{E}[\langle M, M\rangle_\infty] = C' < \infty. \tag{2}$$

Let $n \to \infty$ and use Fatou's lemma to see that $\mathbb{E}[M_t^2] \leq C' < \infty$, so $(M_t)_{t\geq 0}$ is bounded in L^2.

We still have to check that $(M_t)_{t\geq 0}$ is a martingale. However, (2) shows that $(M_{t\wedge T_n})_{n\geq 1}$ is uniformly integrable and so converges both almost surely and in L^1 to M_t for every $t \geq 0$. Recalling that M^{T_n} is a martingale, L^1 convergence implies, for $s > t$,

$$M_t = \lim_n M_t^{T_n} = \lim_n \mathbb{E}[M_s^{T_n} | \mathscr{F}_t] = \mathbb{E}[\lim_n M_s^{T_n} | \mathscr{F}_t] = \mathbb{E}[M_s | \mathscr{F}_t]$$

so M is a martingale.

Finally, if the two properties hold, then $M^2 - \langle M, M \rangle$ is dominated by $\sup_{t\geq 0} M_t^2 + \langle M, M \rangle_\infty$, which is integrable, and so Proposition 7.17 again says that $M^2 - \langle M, M \rangle$ is a uniformly integrable martingale. $\square$

Our previous theorem immediately yields that for a local martingale M with $M_0 = 0$, if $\mathbb{E}[\langle M \rangle_\infty] < \infty$ then

$$\|M\|^2_{\mathscr{H}^2} = \mathbb{E}[M^2_\infty] = \mathbb{E}[\langle M \rangle_\infty].$$

We can also extend our result on finite variation martingales to consider quadratic variation.

Corollary

Let M be a continuous local martingale with $M_0 = 0$. Then the following are equivalent:

1. *M is indistinguishable from zero,*
2. *$\langle M \rangle_t = 0$ for all $t \geq 0$ a.s.,*
3. *M is a process of finite variation.*

Proof.

We already know that the first and third statements are equivalent.

That the first implies the second is trivial, so we must just show that the second implies the first.

We have $\langle M \rangle_\infty = \lim_{t\to\infty} \langle M \rangle_t = 0$. From our characterization of $\mathscr{H}^2$ in terms of quadratic variation, $M \in \mathscr{H}^2$ and $\mathbb{E}[M_\infty^2] = \mathbb{E}[\langle M \rangle_\infty] = 0$ and so $M_t = \mathbb{E}[M_\infty | \mathscr{F}_t] = 0$ almost surely. $\square$