B8.2 Continuous Martingales and Stochastic Calculus

Stochastic Integration

Stochastic integration with respect to continuous local martingales

Samuel Cohen

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Oxford Mathematics

UNIVERSITY OF OXFORD

Mathematical Institute

Definition

For a continuous local martingale M, denote by $\mathrm{L}^2_{\mathrm{loc}}(M)$ the space of progressively measurable processes K such that

$$\forall t \geq 0 \quad \int_0^t K_s^2 \mathrm{d}\langle M\rangle_s < +\infty \ a.s.$$

Theorem

Let M be a continuous local martingale. For any $K \in \mathrm{L}^2_{loc}(M)$ there exists a unique continuous local martingale, zero in zero, denoted $K \bullet M$ and called the Itô integral of K with respect to M, such that for any continuous local martingale N

$$\langle K \bullet M, N \rangle = K \cdot \langle M, N \rangle. \tag{1}$$

If $M \in \mathscr{H}^{2,c}$ and $K \in \mathrm{L}^2(M)$ then this definition coincides with the previous one.

Proof

We only sketch the proof. Not surprisingly, we use a stopping argument.
For every $n \geq 1$, set

$$\tau_n = \inf\left\{ t \geq 0 : \int_0^t (1 + K_s^2)\mathrm{d}\langle M\rangle_s \geq n \right\},$$

so that τ_n is a sequence of stopping times that increases to infinity. Since $\langle M^{\tau_n}\rangle_\infty = \langle M\rangle_{\tau_n} \leq n$, the stopped martingale M^{τ_n} is in $\mathscr{H}^{2,c}$. Also

$$\int_0^\infty K_s^2 d\langle M^{\tau_n}, M^{\tau_n}\rangle_s = \int_0^{\tau_n} K_s^2 d\langle M, M\rangle_s \leq n,$$

so that $K \in \mathrm{L}^2(M^{\tau_n})$ and the definition of $K \bullet M^{\tau_n}$ makes sense.

If $m > n$,

$$K \bullet M^{\tau_n} = (K \bullet M^{\tau_m})^{\tau_n}$$

so there is a unique process, that we denote $K \bullet M$ such that

$$(K \bullet M)^{\tau_n} = K \bullet M^{\tau_n}$$

and $(K \bullet M)_t = \lim_{n\to\infty}(K \bullet M^{\tau_n})_t$ and so, since $(K \bullet M^{\tau_n})$ is a martingale, the process $K \bullet M$ is a continuous local martingale with reducing sequence τ_n.

If N is a continuous local martingale (and without loss of generality $N_0 = 0$), we consider a reducing sequence

$$\tilde{\tau}_n = \inf\{t \geq 0 : |N_t| \geq n\} \quad \text{and set} \quad \rho_n := \tau_n \wedge \tilde{\tau}_n.$$

Then $N^{\rho_n} \in \mathscr{H}_0^{2,c}$ and hence

$$\begin{aligned}
\langle K \bullet M, N\rangle^{\rho_n} &= \langle (K \bullet M)^{\rho_n}, N^{\rho_n}\rangle = \langle (K \bullet M^{\tau_n})^{\rho_n}, N^{\rho_n}\rangle \\
&= \langle K \bullet M^{\tau_n}, N^{\rho_n}\rangle = K \cdot \langle M^{\tau_n}, N^{\rho_n}\rangle = K \cdot \langle M, N\rangle^{\rho_n} \\
&= (K \cdot \langle M, N\rangle)^{\rho_n},
\end{aligned}$$

so that $\langle K \bullet M, N\rangle = K \cdot \langle M, N\rangle$ as required. Uniqueness of $K \bullet M$ follows from the intrinsic characterization. $\square$