B8.2 Continuous Martingales and Stochastic Calculus

Stochastic Integration

Properties of the stochastic integral

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We could write our intrinsic description of the integral as

$$\left\langle \int_0^{\cdot} K_s dM_s, N \right\rangle_t = \int_0^t K_s d\langle M, N\rangle_s \,;$$

that is, the stochastic integral 'commutes' with the bracket. One important consequence is that if $M \in \mathscr{H}^{2,c}$ and $K \in L^2(M)$, then applying the characterization twice gives

$$\langle K \bullet M, K \bullet M\rangle = K \cdot \Big(K \cdot \langle M, M\rangle\Big) = K^2 \cdot \langle M, M\rangle.$$

In other words, the bracket process of $\int K_s dM_s$ is $\int K_s^2 d\langle M, M\rangle_s$. More generally, for N another martingale and $H \in L^2(N)$,

$$\left\langle \int_0^{\cdot} H_s dN_s, \int_0^{\cdot} K_s dM_s \right\rangle_t = \int_0^t H_s K_s d\langle M, N\rangle_s.$$

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Proposition (Associativity of stochastic integration)

Let $H \in L^2(M)$. If K is progressive, then $KH \in L^2(M)$ if and only if $K \in L^2(H \bullet M)$. In that case,

$$(KH) \bullet M = K \bullet (H \bullet M).$$

(This is the analogue of what we already know for finite variation processes, where $K \cdot (H \cdot A) = (KH) \cdot A$.)

Proof.

$$\mathbb{E}\left[\int_0^\infty K_s^2 H_s^2 d\langle M,M\rangle_s\right] = \mathbb{E}\left[\int_0^\infty K_s^2 d\langle H\bullet M, H\bullet M\rangle_s\right],$$

which gives the first assertion.
For the second, for $N \in \mathscr{H}^{2,c}$ we write

$$\begin{aligned}\langle (KH)\bullet M, N\rangle &= KH\cdot\langle M,N\rangle = K\cdot(H\cdot\langle M,N\rangle)\\ &= K\cdot\langle H\bullet M, N\rangle = \langle K\bullet(H\bullet M), N\rangle,\end{aligned}$$

and by uniqueness in our characterization this implies

$$(KH)\bullet M = K\bullet(H\bullet M).$$

□

Recall that if $M \in \mathscr{H}^{2,c}$ and τ is a stopping time, then $M^\tau = (M_{t\wedge\tau}, t \geq 0)$ denotes the stopped process, which is itself a martingale and clearly $M^\tau \in \mathscr{H}^{2,c}$.

For any $N \in \mathscr{H}^{2,c}$ we have

$$\langle M^\tau, N\rangle = \langle M, N\rangle^\tau = 1_{[0,\tau]} \cdot \langle M, N\rangle = \langle 1_{[0,\tau]} \bullet M, N\rangle,$$

so by uniqueness, $1_{[0,\tau]} \bullet M = M^\tau$.

In fact a much more general property holds true.

Proposition (Stopped stochastic integrals)

Let $M \in \mathscr{H}^{2,c}$, $K \in \mathrm{L}^2(M)$ *and* τ *be a stopping time. Then*

$$(K \bullet M)^\tau = K \bullet M^\tau = K 1_{[0,\tau]} \bullet M.$$

Proof.

We already argued above that the result holds for $K \equiv 1$. Associativity says

$$K \bullet M^\tau = K \bullet \left(1_{[0,\tau]} \bullet M\right) = K 1_{[0,\tau]} \bullet M.$$

Applying the same result to the martingale $K \bullet M$ we obtain

$$(K \bullet M)^\tau = 1_{[0,\tau]} \bullet (K \bullet M) = 1_{[0,\tau]} K \bullet M,$$

which gives the desired equalities. □