B8.2 Continuous Martingales and Stochastic Calculus

Stochastic Integration

Stochastic integration with respect to continuous semimartingales

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- Naturally, we're going to define an integral with respect to a continuous semimartingale $X = X_0 + M + A$ as a sum of integrals w.r.t. M and w.r.t. A.
- We need to define the 'right' set of processes for which the integral can be defined.

Definition

Let $X = X_0 + M + A$ be a continuous semimartingale. The space of *X-stochastically integrable processes* is given by

$$\mathrm{L}(X) := \mathrm{L}^2_{\mathrm{loc}}(M) \cap \mathrm{L}^1_{\mathrm{loc}}(|dA|),$$

thart is $K \in \mathrm{L}(X)$ if there are stopping times $\tau_n \to \infty$ such that

$$\mathbb{E}\left[\int_0^{\tau_n} K_t^2 d\langle M\rangle_t\right] < \infty \qquad \text{and} \qquad \int_0^{\tau_n} |K_t||dA_t| < \infty \quad \text{a.s.}$$

A subset of such integrands, particularly convenient since it does not depend on X, is given by

Definition

We say that a progressively measurable process K is locally bounded if

$$\sup_{u \leq t} |K_u| < +\infty \quad \forall t \geq 0, \quad a.s.$$

(As an exercise, check that this agrees with the definition of ‘locally bounded’ given in section 4.3!)

In particular, any adapted process with continuous sample paths is locally bounded.

Proposition

If K is progressively measurable and locally bounded, then it is in $L(X)$ for every continuous semimartingale X.

Proof.

Take $\tau_n = \inf\left\{T : \int_0^T K_t^2 d\langle M\rangle_t + \int_0^T |K_t||dA_t| \geq n\right\}$. $\square$

Definition

Let $X = X_0 + M + A$ be a continuous semimartingale and $K \in L(X)$. The *Itô stochastic integral* of K with respect to X is the continuous semimartingale $K \bullet X$ defined by

$$K \bullet X := K \bullet M + K \cdot A$$

often written

$$(K \bullet X)_t = \int_0^t K_s \mathrm{d} X_s = \int_0^t K_s \mathrm{d} M_s + \int_0^t K_s \mathrm{d} A_s .$$

This integral inherits all the nice properties of the Stieltjes integral and the Itô integral that we have already derived (linearity, associativity, stopping etc.).

And of course, it is still the case for an elementary function $\varphi \in \mathscr{E}$ that

$$(\varphi \bullet X)_t = \sum_{i=1}^{m} \varphi^{(i)} \left(X_{t_{i+1} \wedge t} - X_{t_i \wedge t} \right).$$