B8.2 Continuous Martingales and Stochastic Calculus

Brownian Motion

Small time behaviour of Brownian motion

Samuel Cohen
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UNIVERSITY OF OXFORD
Mathematical Institute

Oxford Mathematics

To study the very small time behaviour of Brownian motion, it is useful to establish the following $0-1$ law.

Theorem (Blumenthal's 0-1 law)

Fix a Brownian motion $(B_t)_{t\geq 0}$ *on* $(\Omega, \mathscr{F}, \mathbb{P})$. *Recall* $B_0 = 0$.

For every $t \geq 0$ *we set* $\mathscr{F}_t := \sigma(B_u : u \leq t)$, *so that* $\mathscr{F}_s \subset \mathscr{F}_t$ *if* $s \leq t$. *We also set* $\mathscr{F}_{0+} := \cap_{s>0} \mathscr{F}_s$.

Then the σ*-field* $\mathscr{F}_{0+}$ *is trivial in the sense that* $\mathbb{P}[A] = 0$ *or* 1 *for every* $A \in \mathscr{F}_{0+}$.

Proof

Let $0 < t_1 < t_2 \cdots < t_k$ and let $g : \mathbb{R}^k \to \mathbb{R}$ be a bounded continuous function. Also, fix $A \in \mathscr{F}_{0+}$. Then by continuity and dominated convergence

$$\mathbb{E}[1_A g(B_{t_1}, \ldots, B_{t_k})] = \lim_{\varepsilon \downarrow 0} \mathbb{E}[1_A g(B_{t_1} - B_\varepsilon, \ldots, B_{t_k} - B_\varepsilon)].$$

If $0 < \varepsilon < t_1$, the variables $B_{t_1} - B_\varepsilon, \ldots, B_{t_k} - B_\varepsilon$ are independent of $\mathscr{F}_\varepsilon$ (by the Markov property) and thus also of $\mathscr{F}_{0+}$.

It follows that

$$\begin{aligned}\mathbb{E}[1_A g(B_{t_1},\ldots,B_{t_k})] &= \lim_{\varepsilon\downarrow 0}\mathbb{E}[1_A g(B_{t_1}-B_\varepsilon,\ldots,B_{t_k}-B_\varepsilon)]\\ &= \mathbb{P}[A]\mathbb{E}[g(B_{t_1},\ldots,B_{t_k})].\end{aligned}$$

We have thus obtained that $\mathscr{F}_{0+}$ is independent of $\sigma(B_{t_1},\ldots,B_{t_k})$. Since this holds for any finite collection $\{t_1,\ldots,t_k\}$ of (strictly) positive reals, $\mathscr{F}_{0+}$ is independent of $\sigma(B_t, t>0)$.

However, $\sigma(B_t, t>0)=\sigma(B_t, t\geq 0)$, since B_0 is the pointwise limit of B_t when $t\to 0$. Since $\mathscr{F}_{0+}\subset\sigma(B_t, t\geq 0)$, we conclude that $\mathscr{F}_{0+}$ is independent of itself and so must be trivial. $\square$

Proposition

Let B be a standard real-valued Brownian motion, as above.

1. *Then, a.s., for every $\varepsilon > 0$,*

$$\sup_{0 \le s \le \varepsilon} B_s > 0 \quad \textit{and} \quad \inf_{0 \le s \le \varepsilon} B_s < 0.$$

   In particular, $\inf\{t > 0 : B_t = 0\} = 0$ a.s.

2. *For every $a \in \mathbb{R}$, let $T_a := \inf\{t \ge 0 : B_t = a\}$ (with the convention that $\inf \emptyset = \infty$). Then a.s. for each $a \in \mathbb{R}$, $T_a < \infty$. Consequently, we have a.s.*

$$\limsup_{t \to \infty} B_t = +\infty, \quad \liminf_{t \to \infty} B_t = -\infty.$$

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Remark

It is not a priori obvious that $\sup_{0\leq s\leq \varepsilon} B_s$ *is even measurable, since this is an uncountable supremum of random variables.*

Since sample paths are continuous, we can restrict to rational *values of* $s \in [0, \varepsilon]$ *so that we are taking the supremum over a countable set. We implicitly use this observation in what follows.*

Proof

(i) Let ε_p be a sequence of strictly positive reals decreasing to zero and set $A := \bigcap_{p\geq 0}\{\sup_{0\leq s\leq \varepsilon_p} B_s > 0\}$. Since this is a monotone decreasing intersection, $A \in \mathscr{F}_{0+}$. On the other hand, by monotonicity,

$$\mathbb{P}[A] = \lim_{p\to\infty}{}^{\downarrow}\Big\{\mathbb{P}[\sup_{0\leq s\leq \varepsilon_p} B_s > 0]\Big\},$$

where $\lim^{\downarrow}$ denotes a decreasing limit, and

$$\mathbb{P}[\sup_{0\leq s\leq \varepsilon_p} B_s > 0] \geq \mathbb{P}[B_{\varepsilon_p} > 0] = \frac{1}{2}.$$

So $\mathbb{P}[A] \geq 1/2$ and by Blumenthal's 0-1 law $\mathbb{P}[A] = 1$. Hence a.s. for all $\varepsilon > 0$, $\sup_{0\leq s\leq \varepsilon} B_s > 0$. Replacing B by $-B$ we obtain $\mathbb{P}[\inf_{0\leq s\leq \varepsilon} B_s < 0] = 1$.

(ii) Write

$$1 = \mathbb{P}\Big[\sup_{0\le s\le 1} B_s > 0\Big] = \lim_{\delta\downarrow 0}{}^{\uparrow}\mathbb{P}\Big[\sup_{0\le s\le 1} B_s > \delta\Big].$$

Now use the scale invariance property, that is $B^{\lambda}_t = B_{\lambda^2 t}/\lambda$ is a Brownian motion, with $\lambda = 1/\delta$ to see that for any $\delta > 0$,

$$\mathbb{P}\Big[\sup_{0\le s\le 1} B_s > \delta\Big] = \mathbb{P}\Big[\sup_{0\le s\le 1/\delta^2} B^{(\delta)}_s > 1\Big] = \mathbb{P}\Big[\sup_{0\le s\le 1/\delta^2} B_s > 1\Big]. \tag{1}$$

If we let $\delta \downarrow 0$, we find

$$\mathbb{P}[\sup_{s\ge 0} B_s > 1] = \lim_{\delta\downarrow 0}{}^{\uparrow}\mathbb{P}[\sup_{0\le s\le 1/\delta^2} B_s > \delta] = 1.$$

Another scaling argument shows that, for every $M > 0$,

$$\mathbb{P}[\sup_{s\geq 0} B_s > M] = 1$$

and replacing B with $-B$,

$$\mathbb{P}[\inf_{s\geq 0} B_s < -M] = 1.$$

Continuity of sample paths completes the proof of (ii). $\square$

Corollary

The map $t \mapsto B_t$ is a.s. not monotone on any non-trivial interval.