B8.2 Continuous Martingales and Stochastic Calculus

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(Sub/super-)Martingales in continuous time

Definitions

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The results in this section will to a large extent mirror what you proved last term for discrete parameter martingales (and we use those results repeatedly in our proofs).

We assume throughout that a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ is given.

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Definition

An adapted stochastic process $(X_t)_{t\geq 0}$ such that $X_t \in \mathrm{L}^1(\mathbb{P})$ (i.e. $\mathbb{E}[|X_t|] < \infty$) for any $t \geq 0$, is called

1. a *martingale* if $\mathbb{E}[X_t|\mathscr{F}_s] = X_s$ for all $0 \leq s \leq t$,
2. a *super-martingale* if $\mathbb{E}[X_t|\mathscr{F}_s] \leq X_s$ for all $0 \leq s \leq t$,
3. a *sub-martingale* if $\mathbb{E}[X_t|\mathscr{F}_s] \geq X_s$ for all $0 \leq s \leq t$.

Exercises: Suppose $(Z_t : t \geq 0)$ is an adapted process with independent increments, i.e. for all $0 \leq s < t$, $Z_t - Z_s$ is independent of $\mathscr{F}_s$. The following give us examples of martingales:

1. if $\forall\, t \geq 0$, $Z_t \in \mathrm{L}^1$, then $\tilde{Z}_t := Z_t - \mathbb{E}[Z_t]$ is a martingale,
2. if $\forall\, t \geq 0$, $Z_t \in \mathrm{L}^2$, then $\tilde{Z}_t^2 - \mathbb{E}[\tilde{Z}_t^2]$ is a martingale,
3. if for some $\theta \in \mathbb{R}$, and $\forall\, t \geq 0$, $\mathbb{E}[e^{\theta Z_t}] < \infty$, then $\frac{e^{\theta Z_t}}{\mathbb{E}[e^{\theta Z_t}]}$ is a martingale.

In particular, B_t, $B_t^2 - t$ and $e^{\theta B_t - \theta^2 t/2}$ are all martingales with respect to a filtration $\{\mathscr{F}_t\}_{t \geq 0}$ for which $(B_t)_{t \geq 0}$ is a Brownian motion.

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Warning: It is important to remember that a process is a martingale *with respect to a filtration* – giving yourself more information (enlarging the filtration) may destroy the martingale property.

For us, even when we don't explicitly mention it, there is a filtration implicitly assumed (usually the natural filtration associated with the process, augmented to satisfy the usual conditions).

Given a martingale (or submartingale) it is easy to generate many more.

Proposition

Let $(X_t)_{t\geq 0}$ *be a martingale (respectively sub-martingale) and* $\varphi : \mathbb{R} \to \mathbb{R}$ *be a convex (respectively convex and increasing) such that* $\mathbb{E}[|\varphi(X_t)|] < \infty$ *for any* $t \geq 0$*. Then* $(\varphi(X_t))_{t\geq 0}$ *is a sub-martingale.*

Proof.

Apply the conditional Jensen inequality (see appendix). $\square$

In particular, if $(X_t)_{t\geq 0}$ is martingale with $\mathbb{E}[|X_t|^p] < \infty$, for some $p \geq 1$ and all $t \geq 0$, then $|X_t|^p$ is a sub-martingale (and consequently, $t \mapsto \mathbb{E}[|X_t|^p]$ is non-decreasing).