B8.2 Continuous Martingales and Stochastic Calculus

Strong Markov property and reflection principle

The reflection principle

Samuel Cohen
Hilary Term 2021

UNIVERSITY OF OXFORD
Mathematical Institute

Oxford Mathematics

- The following result, known as the reflection principle, was known at the end of the 19th Century for random walk.
- It appears in the famous 1900 thesis of Bachelier, which introduced the idea of modelling stock prices using Brownian motion (although since he had no formulation of the strong Markov property, his proof is not rigorous).

Theorem (The reflection principle)

Let $S_t := \sup_{u \leq t} B_u$*. For* $a \geq 0$ *and* $b \leq a$ *we have*

$$\mathbb{P}[S_t \geq a, B_t \leq b] = \mathbb{P}[B_t \geq 2a - b] \quad \forall t \geq 0.$$

In particular S_t *and* $|B_t|$ *have the same distribution.*

Proof

We apply the strong Markov property to the stopping time $T_a = \inf\{t > 0 : B_t = a\}$. We have already seen that $T_a < \infty$ a.s. and so in the notation used when stating the strong Markov property,

$$\mathbb{P}[S_t \geq a, B_t \leq b] = \mathbb{P}[T_a \leq t, B_t \leq b] = \mathbb{P}[T_a \leq t, B_{t-T_a}^{(T_a)} \leq b - a] \tag{1}$$

(since $B_{t-T_a}^{(T_a)} = B_t - B_{T_a} = B_t - a$).

Now $B^{(T_a)}$ is a Brownian motion, independent of $\mathscr{F}_{T_a}$ and hence of T_a. Since $B^{(T_a)}$ has the same distribution as $-B^{(T_a)}$, $(T_a, B^{(T_a)})$ has the same distribution as $(T_a, -B^{(T_a)})$.

So

$$\begin{aligned}
&\mathbb{P}[T_a \le t, B^{(T_a)} \le b - a] \\
&= \mathbb{P}[T_a \le t, -B^{(T_a)}_{t-T_a} \le b - a] \\
&= \mathbb{P}[T_a \le t, B_t \ge 2a - b] = \mathbb{P}[B_t \ge 2a - b],
\end{aligned}$$

since $2a - b \ge a$ and so $\{B_t \ge 2a - b\} \subseteq \{T_a \le t\}$.

We have proved that $\mathbb{P}[S_t \ge a, B_t \le b] = \mathbb{P}[B_t \ge 2a - b]$. For the last assertion of the theorem, taking $a = b$ in (1), observe that

$$\begin{aligned}
\mathbb{P}[S_t \ge a] &= \mathbb{P}[S_t \ge a, B_t \ge a] + \mathbb{P}[S_t \ge a, B_t \le a] \\
&= 2\mathbb{P}[B_t \ge a] = \mathbb{P}[B_t \ge a] + \mathbb{P}[B_t \le -a] \quad \text{(symmetry)} \\
&= \mathbb{P}[|B_t| \ge a].
\end{aligned}$$

$\square$