B8.2 Continuous Martingales and Stochastic Calculus

Brownian Motion

Fine continuity of Brownian sample paths

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- From now on, when we say “Brownian motion”, we mean a standard real-valued Brownian motion.
- We know that $t \mapsto B_t(\omega)$ is continuous.
- **Exercise:** Use the Kolmogorov continuity criterion to show that Brownian motion admits a modification which is locally Hölder continuous of order γ for any $0 < \gamma < 1/2$.
- On the other hand, as we have already remarked, the path is actually rather ‘rough’. We’d like to have a way to quantify this roughness.

Definition

Let π be a partition of $[0, T]$, $N(\pi)$ the number of intervals that make up π and $\|\pi\|$ be the *mesh* of π (that is the length of the longest interval in the partition). Write
$0 = t_0 < t_1 < \ldots < t_{N(\pi)} = T$ for the endpoints of the intervals of the partition. Then the *variation* of a function $f : [0, T] \to \mathbb{R}$ is

$$\lim_{\delta \to 0} \left\{ \sup_{\pi : \|\pi\| = \delta} \sum_{j=1}^{N(\pi)} |f(t_j) - f(t_{j-1})| \right\}.$$

If the function is 'nice', for example differentiable, then it has bounded variation. Our 'rough' paths will have *unbounded* variation.

To quantify roughness we can extend the idea of variation to that of p-variation.

Definition

In the notation of Definition 1, the p-variation of a function $f : [0, T] \to \mathbb{R}$ is defined as

$$\lim_{\delta \to 0} \left\{ \sup_{\pi : \|\pi\| = \delta} \sum_{j=1}^{N(\pi)} |f(t_j) - f(t_{j-1})|^p \right\}.$$

Notice that for $p > 1$, the p-variation will be finite for functions that are much rougher than those for which the variation is bounded. For example, roughly speaking, finite 2-variation will follow if the fluctuation of the function over an interval of order δ is order $\sqrt{\delta}$.

For a typical Brownian path, the 2-variation will be infinite. However, a slightly weaker analogue of the 2-variation *does* exist.

Theorem

Let B_t denote Brownian motion under $\mathbb{P}$ and for a partition π of $[0, T]$ define

$$S(\pi) = \sum_{j=1}^{N(\pi)} \left| B_{t_j} - B_{t_{j-1}} \right|^2.$$

Let π_n be a sequence of partitions with $\|\pi_n\| \to 0$. Then

$$\mathbb{E}\left[\left| S(\pi_n) - T \right|^2 \right] \to 0 \qquad \text{as } n \to \infty. \tag{1}$$

We say that the quadratic variation process of Brownian motion, which we denote by $\{\langle B\rangle_t\}_{t\geq 0}$ is $\langle B\rangle_t = t$. More generally, we can define the quadratic variation process associated with any bounded continuous martingale.

Definition

Suppose that $\{M_t\}_{t\geq 0}$ is a bounded continuous $\mathbb{P}$-martingale. The *quadratic variation* process associated with $\{M_t\}_{t\geq 0}$ is the process $\{\langle M\rangle_t\}_{t\geq 0}$ such that for any sequence of partitions π_n of $[0,T]$ with $\|\pi_n\|\to 0$,

$$\mathbb{E}\left[\left|\sum_{j=1}^{N(\pi_n)}\left|M_{t_j}-M_{t_{j-1}}\right|^2-\langle M\rangle_T\right|^2\right]\to 0 \qquad \text{as } n\to\infty. \tag{2}$$

Proof of Theorem

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We expand the expression inside the expectation in (1) and make use of our knowledge of the normal distribution.

Let $\{t_{n,j}\}_{j=0}^{N(\pi_n)}$ denote the endpoints of the intervals that make up the partition π_n.
First observe that

$$|S(\pi_n) - T|^2 = \left| \sum_{j=1}^{N(\pi_n)} \left\{ \left| B_{t_{n,j}} - B_{t_{n,j-1}} \right|^2 - (t_{n,j} - t_{n,j-1}) \right\} \right|^2 .$$

It is convenient to write $\delta_{n,j}$ for $\left|B_{t_{n,j}} - B_{t_{n,j-1}}\right|^2 - (t_{n,j} - t_{n,j-1})$. Then

$$|S(\pi_n) - T|^2 = \sum_{j=1}^{N(\pi_n)} \left(\delta_{n,j}^2 + 2 \sum_{k>j} \delta_{n,j} \delta_{n,k} \right).$$

Note that since Brownian motion has independent increments,

$$\mathbb{E}[\delta_{n,j}\delta_{n,k}] = \mathbb{E}[\delta_{n,j}]\,\mathbb{E}[\delta_{n,k}] = 0 \qquad \text{if } j \neq k.$$

Also

$$\mathbb{E}[\delta_{n,j}^2] = \mathbb{E}\Big[\left|B_{t_{n,j}} - B_{t_{n,j-1}}\right|^4 - 2\left|B_{t_{n,j}} - B_{t_{n,j-1}}\right|^2 (t_{n,j} - t_{n,j-1}) + (t_{n,j} - t_{n,j-1})^2 \Big].$$

For a normally distributed random variable, X, with mean zero and variance λ, $\mathbb{E}[|X|^4] = 3\lambda^2$, so we have

$$\begin{aligned}
\mathbb{E}\left[\delta_{n,j}^2\right] &= 3\left(t_{n,j} - t_{n,j-1}\right)^2 - 2\left(t_{n,j} - t_{n,j-1}\right)^2 + \left(t_{n,j} - t_{n,j-1}\right)^2 \\
&= 2\left(t_{n,j} - t_{n,j-1}\right)^2 \\
&\leq 2\|\pi_n\|\left(t_{n,j} - t_{n,j-1}\right).
\end{aligned}$$

Summing over j

$$\begin{aligned}
\mathbb{E}\left[\left|S(\pi_n) - T\right|^2\right] &\leq 2\sum_{j=1}^{N(\pi_n)} \|\pi_n\|\left(t_{n,j} - t_{n,j-1}\right) \\
&= 2\|\pi_n\|T \\
&\to 0 \qquad \text{as } n \to \infty.
\end{aligned}$$

□

Corollary

Brownian sample paths are of infinite variation on any interval almost surely.

Corollary

Brownian sample paths are almost surely nowhere locally Hölder continuous of order $\gamma > \frac{1}{2}$.

(The proofs are exercises.)

In fact, a very precise statement is possible.

Theorem (Lévy's modulus of continuity (Not Examinable))

For B a Brownian motion,

$$\limsup_{\varepsilon\downarrow 0}\left(\sup_{0\le s<t\le 1,\, t-s\le\varepsilon}\frac{|B_t-B_s|}{\sqrt{2\varepsilon\log(1/\varepsilon)}}\right)=1 \qquad a.s.$$

Consequently, Brownian sample paths are almost surely nowhere locally Hölder continuous of order $\gamma=1/2$, and the 2-variation is almost surely infinite.

Proof.

Omitted (proof is a straightforward but fiddly calculation of estimates, see, for example, Revuz & Yor, p30ff) $\square$