B8.2 Continuous Martingales and Stochastic Calculus

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Continuous semimartingales

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Definition

A stochastic process $X = (X_t : t \geq 0)$ is called a continuous semimartingale if it can be written as

$$X_t = X_0 + M_t + A_t, \quad t \geq 0 \tag{1}$$

where M is a continuous local martingale, A is a continuous process of finite variation, and $M_0 = A_0 = 0$ a.s..

The decomposition is unique (up to indistinguishability). It should be remembered that there is a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ and a probability measure $\mathbb{P}$ implicit in our definition.

Proposition

A continuous semimartingale is of finite quadratic variation and, in the notation above, $\langle X, X\rangle = \langle M, M\rangle$.

- If X, Y are two continuous semimartingales, we can define their co-variation $\langle X, Y\rangle$ via the polarisation formula that we used for martingales.
- If $X_t = X_0 + M_t + A_t$ and $Y_t = Y_0 + N_t + A'_t$, then $\langle X, Y\rangle_t = \langle M, N\rangle_t$.

Proof.

Fix $t \geq 0$ and consider a sequence of partitions of $[0, t]$, $\pi_m = \{0 = t_0 < t_1 < \ldots < t_{n_m} = t\}$ with $\|\pi_m\| \to 0$ as $m \to \infty$. Then

$$\sum_{i=1}^{n_m}(X_{t_i} - X_{t_{i-1}})^2 = \underbrace{\sum_{i=1}^{n_m}(M_{t_i} - M_{t_{i-1}})^2}_{(i)} + \underbrace{\sum_{i=1}^{n_m}(A_{t_i} - A_{t_{i-1}})^2}_{(ii)}$$

$$+ 2\underbrace{\sum_{i=1}^{n_m}(M_{t_i} - M_{t_{i-1}})(A_{t_i} - A_{t_{i-1}})}_{(iii)}.$$

It follows from the properties of M and A that, as $m \to \infty$,

$$(i) \to \langle M, M\rangle_t,$$

$$(ii) \leq \sup_{1\leq i\leq n_m} |A_{t_i} - A_{t_{i-1}}| \cdot V_t(A) \to 0 \quad \text{a.s. },$$

$$(iii) \leq \sup_{1\leq i\leq n_m} |M_{t_i} - M_{t_{i-1}}| \cdot V_t(A) \to 0 \quad \text{a.s. .}$$