B8.2: Continuous Martingales and Stochastic Calculus (2021)
Problem Sheet 2

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The questions on this sheet are divided into two sections. Those in the first section are compulsory and should be handed in for marking. Those in the second are extra practice questions and should not *be handed in.*

The questions are not in order of difficulty; if you are stuck on one question, move on to the next.

Section 1 (Compulsory)

1. Let $(B_t)_{t\geq 0}$ be a Brownian motion. Show that for every $M > 0$,

$$\mathbb{P}[\sup_{s\geq 0} B_s > M] = 1.$$

2. Consider the following stochastic process

$$X_t := x(1-t) + yt + (B_t - tB_1); 0 \leq t \leq 1.$$

   (a) Show that X is a continuous Gaussian process with $X_0 = x$ and $X_1 = y$.

   (b) Show that X cannot be adapted to $(\mathcal{F}_t)$. Is B also an $(\mathcal{F}_t^X)$-Brownian motion on $[0, 1]$?

   (c) Calculate the mean and covariance function of $(X_t)_{0\leq t\leq 1}$.

   (d) Verify that X_t has the same law as the conditional process $(W_t|W_0 = x, W_1 = y)$, where W is a Brownian motion.

   X is called the Brownian bridge from x to y over $[0, 1]$.

3. Show that if a stochastic process (X_t) adapted to a filtration $(\mathcal{F}_t)$ has

   (a) right-continuous paths then for an open set Γ, $H_\Gamma := \inf\{t \geq 0 : X_t \in \Gamma\}$ is a stopping time relative to $(\mathcal{F}_{t+})$,

   (b) continuous paths then for a closed set Γ, H_Γ is a stopping time relative to $(\mathcal{F}_t)$.

4. Let τ and ρ be two stopping times relative to a given filtration $(\mathcal{F}_t)$. Show that

   (a) $\tau \wedge \rho := \min\{\tau, \rho\}$, $\tau \vee \rho := \max\{\tau, \rho\}$ and $\tau + \rho$ are all also stopping times

   (b) $\mathcal{F}_{\tau\wedge\rho} = \mathcal{F}_\tau \cap \mathcal{F}_\rho$ and $\{\tau \leq \rho\}$ is $\mathcal{F}_{\tau\wedge\rho}$ measurable

5. Let H_a be the first hitting time of a, $H_a = \inf\{t \geq 0 : B_t = a\}$. (We retain this notation for the next two questions.)

   Use Optional Stopping to compute the distribution of $B_{H_a\wedge H_b}$ for $a < 0 < b$.

6. Recall that in lectures we used Optional Stopping to show that the Laplace transform of H_a is given by

$$\mathbb{E}\left[e^{-\lambda H_a}\right] = e^{-|a|\sqrt{2\lambda}} \quad a \in \mathbb{R}, \lambda > 0.$$

Consider $a, b > 0$. Deduce that if ξ_a, ξ_b are independent and distributed as H_a and H_b respectively, then $\xi_a + \xi_b$ has the same distribution as H_{a+b}. Use the strong Markov property to find an alternative proof of this result.

7. Use Optional Stopping to show that the Laplace transform of $H_a \wedge H_{-a}$ is given by

$$\mathbb{E}\left[e^{-\lambda H_a \wedge H_{-a}}\right] = \frac{1}{\cosh(a\sqrt{2\lambda})}, \quad a > 0, \lambda > 0.$$

Hint: use martingales $(M^{(\theta)} + M^{(-\theta)})/2$, *where* $M_t^{(\theta)} := \exp(\theta B_t - \theta^2 t/2)$.

Section 2 (Extra practice questions, not for hand-in)

A. Without reference to Lévy's modulus of continuity, show that Brownian sample paths are almost surely of infinite γ-variation for any $\gamma \in [1, 2)$.

B. Fix $t > 0$. Without reference to Lévy's modulus of continuity, show that a.s. Brownian motion is not differentiable with respect to t at time t.
Hint:

- *Argue that* $|B_{t+\epsilon} - B_t|/\epsilon$ *diverges to* $+\infty$ *with probability* 1 *as* $\epsilon \to 0$.
- *Recall that by Blumenthal's 0-1 law for any* $\epsilon > 0$

$$\sup_{0 \le u \le \epsilon} B_{t+u} - B_t > 0 \quad \textit{and} \quad \inf_{0 \le u \le \epsilon} B_{t+u} - B_t < 0 \ \textit{a.s.}$$

- *Draw conclusions on a.s. behaviour of* $\limsup$ *and* $\liminf$ *of* $(B_{t+\epsilon} - B_t)/\epsilon$ *with* $\epsilon \to 0$.

In fact a stronger property holds: a.s. the sample paths are nowhere differentiable

C. Let τ be a stopping time relative to a given filtration $(\mathcal{F}_t)$. Show that $\mathcal{F}_\tau$ is a σ-algebra and τ is $\mathcal{F}_\tau$ measurable.

D. Let τ be a stopping time relative to a given filtration $(\mathcal{F}_t)$. We write

$$\begin{aligned}
\mathcal{F}_\tau &:= \{A \in \mathcal{F} \,:\, A \cap \{\tau \le t\} \in \mathcal{F}_t \,\forall t \ge 0\}, \\
\mathcal{F}_{\tau+} &:= \{A \in \mathcal{F} \,:\, A \cap \{\tau < t\} \in \mathcal{F}_t\}, \\
\mathcal{F}_{\tau-} &:= \sigma(\{A \cap \{\tau > t\} \,:\, t \ge 0,\, A \in \mathcal{F}_t\})
\end{aligned}$$

Show that

(a) $\mathcal{F}_\tau \subset \mathcal{F}_{\tau+}$ and $\mathcal{F}_\tau = \mathcal{F}_{\tau+}$ if $(\mathcal{F}_t)$ is right-continuous;

(b) τ is $\mathcal{F}_{\tau-}$-measurable;

(c) if $\tau = t$ is deterministic then $\mathcal{F}_\tau = \mathcal{F}_t$ and $\mathcal{F}_{\tau+} = \mathcal{F}_{t+}$;

(d) if τ_n is a non-decreasing sequence of stopping times (i.e. for any $\omega \in \Omega$ and $n < m$ $\tau_n(\omega) \le \tau_m(\omega)$) then $\tau := \lim_{n\to\infty} \tau_n$ is also a stopping time;

(e) if τ_n is a non-increasing sequence of stopping times then $\tau := \lim_{n\to\infty} \tau_n$ is a stopping time relative to $(\mathcal{F}_{t+})$.

E. Let $S_t := \sup_{0\le u\le t} B_u$. Deduce from the reflection principle that the couple (S_t, B_t) has density given by

$$\frac{2(2a-b)}{\sqrt{2\pi t^3}} \exp\left(-\frac{(2a-b)^2}{2t}\right) \mathbf{1}_{a>0, b<a}.$$

F. Let $H_a = \inf\{t \ge 0 : B_t = a\}$ be the first hitting time of a.

(a) Show that H_a has the same distribution as $\frac{a^2}{B_1^2}$ and deduce its density.

(b) Using the strong Markov property show that for any continuous bounded function f we have

$$\mathbb{E}[f(H_b - H_a)|\mathcal{F}_{H_a}] = \mathbb{E}[f(H_{b-a})], \quad 0 \le a \le b.$$

Deduce that $(H_a)_{a\ge 0}$ has stationary and independent increments. Discuss the properties of its paths $a \to H_a(\omega)$.

G. Suppose that $(Z_t : t \ge 0)$, defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge0}, \mathbb{P})$, is adapted and has independent increments i.e. for any $0 \le s < t$, $Z_t - Z_s$ is independent of $\mathcal{F}_s$. Show that

- if $\mathbb{E}[|Z_t|] < \infty$ for all $t \ge 0$, then $\tilde{Z}_t := Z_t - \mathbb{E}[Z_t]$ is an $(\mathcal{F}_t)$–martingale;
- if $\mathbb{E}[Z_t^2] < \infty$ for all $t \ge 0$, then $\tilde{Z}_t^2 - \mathbb{E}[\tilde{Z}_t^2]$ is an $(\mathcal{F}_t)$–martingale;
- if for some $\theta \in \mathbb{R}$, $\mathbb{E}\left[e^{\theta Z_t}\right] < \infty$ for all $t \ge 0$, then $\frac{\exp(\theta Z_t)}{\mathbb{E}[\exp(\theta Z_t)]}$ is an $(\mathcal{F}_t)$–martingale.

H. Let $\mathcal{Z}(\omega) := \{t : B_t(\omega) = 0\}$ be the set of Brownian zeros. Argue that $\mathcal{Z}$ is a closed set. Using Fubini's theorem show that the Lebesgue measure of $\mathcal{Z}$ is zero a.s.

Let $R_t = \inf\{u \ge t : B_u = 0\}$. Using the strong Markov property and known facts about Brownian paths show that for any $t \ge 0$

$$\mathbb{P}[\inf\{u > 0 : B_{R_t+u} = 0\} > 0] = 0$$

and deduce that

$$\mathbb{P}\left[\inf\{u > 0 : B_{R_t+u} = 0\} > 0 \text{ for some rational } t\right] = 0.$$

Conclude that a.s. if a point $t \in \mathcal{Z}(\omega)$ is isolated from left, i.e. $(q,t) \cap \mathcal{Z}(\omega) = \emptyset$ for some rational $q < t$ then necessarily t is a decreasing limit of points in $\mathcal{Z}(\omega)$ and thus $\mathcal{Z}$ does not have isolated points.
Fact: It follows that a.s. $\mathcal{Z}$ is uncountable, as it is a 'perfect set', ie. it is a non-empty closed set with no isolated points.