B8.2: Continuous Martingales and Stochastic Calculus (2021)
Problem Sheet 4

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The questions on this sheet are divided into two sections. Those in the first section are compulsory and should be handed in for marking. Those in the second are extra practice questions and should not *be handed in.*

The questions are not in order of difficulty; if you are stuck on one question, move on to the next.

Section 1 (Compulsory)

1. Let $M, N \in \mathcal{H}^{2,c}$, $K \in L^2(M)$ and $F \in L^2(N)$. Show that for each $t \in [0, \infty]$ we have

$$\mathbb{E}\left[\left(\int_0^t K_s dM_s\right)\left(\int_0^t F_s dN_s\right)\right] = \mathbb{E}\left[\int_0^t K_s F_s d\langle M, N\rangle_s\right].$$

2. Suppose that M is a continuous local martingale and $K \in L^2_{\mathrm{loc}}(M)$. Fix $t > 0$. Show that if $\mathbb{E}\left[\int_0^t K_s^2 d\langle M\rangle_s\right] < \infty$ then the stopped process $(K \bullet M)^t$ is a martingale and

$$\mathbb{E}\left[\int_0^t K_s dM_s\right] = 0, \quad \mathbb{E}\left[\left(\int_0^t K_s dM_s\right)^2\right] = \mathbb{E}\left[\int_0^t K_s^2 d\langle M\rangle_s\right] < \infty.$$

3. Let f be a continuous function on $[0, \infty)$ and B a standard Brownian motion. Prove that the random variable

$$X_t := \int_0^t f(s) dB_s, \quad t \geq 0,$$

is Gaussian and compute the covariance of X_t and X_s.

(The same result holds true for locally bounded Borel functions f.)
Hint: You may use that the space of mean-zero Gaussian variables is a closed subspace of L^2.

4. Suppose that $(B_t)_{t\geq 0}$ is standard Brownian motion and f and g are twice continuously differentiably real-valued functions. Using Itô's formula, decompose the semimartingale $X_t = \exp\left(f(B_t) - \int_0^t g(B_s) ds\right)$ into a local martingale and a bounded variation part and hence find an expression relating f and g which guarantees that $(X_t)_{t\geq 0}$ is a local martingale.

5. Let B be a standard Brownian motion. Recall that $M_t^\theta := \exp(\theta B_t - \frac{\theta^2}{2} t)$ is a local martingale. Expanding as a Taylor series in θ, around $\theta = 0$, we can write

$$M_t^\theta = \sum_{k=0}^{\infty} \theta^k H_k(t, B_t),$$

where $H_k(t, x)$ are polynomials.

(a) Find the first four of the $H_k(t,x)$ and show that $(H_k(t,B_t) : t \geq 0)$, $k = 0,1,2,3,4$ are local martingales. *(Hint: you may use the Itô formula to verify the local martingale property)*

(b) We now show that in fact for any local martingale M, $H_k(\langle M\rangle_t, M_t)$ are local martingales and deduce a stochastic integral representation for them. Define h_k via

$$\sum_{k=0}^{\infty} u^k h_k(x) = \exp(ux - u^2/2), \quad u, x \in \mathbb{R}.$$

Let $f(x) = \exp(-x^2/2)$ and deduce that

$$h_k(x) = \frac{(-1)^k}{k! f(x)} f^{(k)}(x).$$

Note that for $a > 0$, we have

$$\exp(ux - au^2/2) = \exp\left(u\sqrt{a}\left(\frac{x}{\sqrt{a}}\right) - \frac{(u\sqrt{a})^2}{2}\right)$$

and deduce that $H_k(a,x) = a^{k/2} h_k(x/\sqrt{a})$. Give the value of $H_k(0,x)$.

(c) Use Itô formula and the above representation to show that if M is a continuous local martingale then $(H_k(\langle M\rangle_t, M_t) : t \geq 0)$ is a continuous local martingale.

(d) Observe that $\frac{\partial H_k}{\partial x}(a,x) = H_{k-1}(a,x)$. Show by induction that

$$H_k(\langle M\rangle_t, M_t) = \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} 1 \, dM_{s_n} \cdots dM_{s_2} \, dM_{s_1}.$$

Section 2 (Extra practice questions, not for hand-in)

A. Use a heuristic argument based on a Taylor expansion to check that for Stratonovich stochastic calculus the chain rule takes the form of the classical (Newtonian) one.

B. (Brownian local time at zero) Let B be a standard Brownian motion. Let $f(x) = |x|$ and f_n be a sequence of convex C^2 functions converging pointwise to $f(x)$ with $f_n'(x)$ increasing in n to $f_-'(x)$ (the left-hand derivative of f which is well defined everywhere).

Such a sequence can be constructed quasi-explicitly. Indeed, take $h(x)$ a non-negative C^∞ function supported on $[-1,0]$ and $\int_{-1}^0 h(x)dx = 1$. Put $f_n(x) := n\int_{-1}^0 f(x+y)h(ny)dy$ and verify it satisfies the announced properties.

Apply Itô's formula to $f_n(B_t)$ and denote L_t^n the finite variation term in the resulting semimartingale decomposition of $f_n(B_t)$. Observe that L_t^n is a non-decreasing process.

(a) Exhibit the region where $f_n''(x)$ is non-zero and hence comment when along the paths of B the process L^n is increasing and when it is constant and deduce what, if it existed, the limit would measure?

(b) Define $sgn(x)$ to be 1 for $x > 0$ and -1 for $x \leq 0$. Use stochastic dominated convergence theorem to show that for any $t > 0$, $\int_0^t f_n'(B_s)dB_s$ converge, in probability and uniformly in $s \leq t$, to $\int_0^t sgn(B_s)dB_s$.

(c) Deduce that L_t^n converges in probability to some process L_t which is non-decreasing and in particular that $|B_t|$ is a semimartingale.
Hint: to deduce monotonicity of L you may want to take a subsequence and pass to a.s. convergence.

(d) Finally, using Itô on B and $|B|$ for a suitable choice of function show that

$$\forall t \geq 0 \quad \int_0^t |B_s| dL_s = 0 \quad a.s.$$

from which you should deduce that L is supported on $\mathcal{Z}$ a.s.
(i.e. for any $s < t$, $L_t(\omega) - L_s(\omega) = \int_s^t dL_u(\omega) = \int_{[s,t]\cap\mathcal{Z}(\omega)} dL_u(\omega)$ a.s.)
The process L is called the local time in zero.

(e) How would you go about defining local time at level a?

(f) Can you see if the above extends to an arbitrary continuous local martingale M?