# B7.3 Further Quantum Theory

## Sheet 1 — HT21

`[Last update: 22:01 on Saturday 16th January, 2021]`

*Throughout this course we will use examples of quantum systems that were introduced in the Part A Quantum Theory course. It will be important to be familiar with these examples, so if you have forgotten some of this, you should solve the revision problems on any problem sheet. Problems listed as "unmarked" are for practice and will not be marked by the TA.*

### 1.0 Harmonic oscillator (revision, unmarked)

1. Consider a quantum particle moving in one dimension. Show that the harmonic oscillator Hamiltonian
$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2 \ ,$$
can be expressed as $H = \hbar\omega\left(\alpha_+\alpha_- + \frac{1}{2}\right) = \hbar\omega\left(\alpha_-\alpha_+ - \frac{1}{2}\right)$ where you should define $\alpha_\pm$ with $(\alpha_+)^* = \alpha_-$.

2. Further show that for any state vector $\psi$,
$$\|\alpha_\pm\psi\|^2 = \frac{\mathbb{E}_\psi(H)}{\hbar\omega} \pm \frac{1}{2} \ .$$

3. Finally, show that
$$[\alpha_-, \alpha_+] = 1 \ , \qquad [H, \alpha_\pm] = \pm\hbar\omega\alpha_\pm \ .$$

4. Suppose there is a normalised state $|\psi_0\rangle$ (the *ground state*) for which $\alpha_-|\psi_0\rangle = 0$. Show that $|\psi_n\rangle = (\alpha_+)^n|\psi_0\rangle$ is an energy eigenstate with energy $E_n = (n + \frac{1}{2})\hbar\omega$.

   Comment on the existence and uniqueness of these energy eigenstates.

### 1.1 Practice with bra-ket notation (unmarked)

Consider the outer product $|\xi\rangle\langle\eta|$, which acts as a linear operator on a Hilbert space $|\xi\rangle\langle\eta| : |\zeta\rangle \mapsto |\xi\rangle\langle\eta|\zeta\rangle = (\eta, \zeta)|\xi\rangle$.

1. Show that if $|\xi\rangle$ is normalised then $|\xi\rangle\langle\xi|$ is an orthogonal projection onto the one-dimensional subspace spanned by $|\xi\rangle$.

2. Demonstrate that if $\phi_1, \phi_2, \ldots \phi_n$ is an orthonormal basis for the Hilbert space $\mathcal{H}$, then $\sum_{j=1}^{n} |\phi_j\rangle\langle\phi_j|$ is the identity operator $\ _{\mathcal{H}}$.

3. Show that the trace of the linear operator $|\xi\rangle\langle\eta|$ is
$$\mathrm{Tr}\Big[|\xi\rangle\langle\eta|\Big] = (\eta, \xi) \ .$$

[*The trace of a linear operator A can be defined as* $\mathrm{Tr}[A] = \sum_j(\phi_j, A\phi_j)$ with $\{\phi_j\}$ an orthonormal basis.]


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### 1.2 Coherent states for the harmonic oscillator

*This problem is a modified version of problem 2.19 from Sakurai's* Modern Quantum Mechanics. *We inherit notation from the previous revision problem.*

1. A *coherent state* of the quantum harmonic oscillator is an eigenstate of the normalized annihilation operator $\alpha_-$. Show that for any $\lambda \in \mathbb{C}$, the state $|\lambda\rangle = e^{-\frac{1}{2}|\lambda|^2} e^{\lambda \alpha_+} |\psi_0\rangle$ is a normalised coherent state with $\alpha_-$ eigenvalue $\lambda$.

2. Determine the expansion of the coherent state $|\lambda\rangle$ in terms of the (normalised) energy eigenstates $|n\rangle = |\psi_n\rangle / \|\psi_n\|$. Find the most probable value of $n$ to be observed when measuring the energy of $|\lambda\rangle$.

3. Show that a coherent state can be obtained by applying the *finite displacement operator* $T(a) = \exp(-iPa/\hbar)$ to the ground state $|\psi_0\rangle$.

4. Determine the overlap between two different coherent states $|\lambda\rangle$ and $|\mu\rangle$ for $\mu \neq \lambda$. Explain why there is no contradiction in the fact that states with different eigenvalues are not orthogonal.

*In this problem it may be useful to look up and use the "Baker-Campbell-Hausdorff" formula if you have not seen it before.*

### 1.3 Dispersion of Gaussian wave packet

Consider a free particle in one dimension whose initial wave function (at time zero) is the stationary *Gaussian wave packet* given by

$$\psi(x) = \frac{1}{(2\pi\sigma^2)^{1/4}} \exp\left(-\frac{x^2}{4\sigma^2}\right) .$$

1. Find the corresponding wave function $\widehat{\psi}(p)$ in momentum space by decomposing the initial wave function into (generalised) momentum eigenstates $|p\rangle$.

2. Now evolve the resulting state to some time $t$ using the fact that the free particle Hamiltonian is diagonalised on the momentum states.

3. Calculate directly the time evolution of the position-space wave function using the free particle propagator $U(x', t'; x, t)$ as described in lectures.

   Comment on the behaviour of the dispersions $\Delta_\psi(X)$ and $\Delta_\psi(P)$ over time.

*In this problem, you may find it useful to know (if you don't know it already) the expression for the general Gaussian integral (valid for $\Re(A) \geqslant 0$) is*

$$\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}Ax^2 + Bx\right) \mathrm{d}x = \sqrt{\frac{2\pi}{A}} \exp\left(\frac{B^2}{2A}\right) .$$

### 1.4 Identical particles

1. Consider a system of two identical particles. First, assume that they are free to move in the interval $[0, a]$ with respective coordinates $x_1$ and $x_2$.

   (a) Explain why, if the particles are fermions, the ground state wave function is

   $$\psi(x_1, x_2) = \frac{1}{\sqrt{2}} \left(\psi_1(x_1)\psi_2(x_2) - \psi_1(x_2)\psi_2(x_1)\right) ,$$

   where $x_1$ and $x_2$ are the positions of the particles and $\psi_k(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{k\pi x}{a}\right)$. What is the ground state wave function if the particles are bosons?

   (b) In both cases, calculate the probability that, in the ground state, both particles will be found in the interval $[0, \frac{1}{2}a]$.

2. Now the particles move on the full $x$-axis with quadratic external potential $V(x) = \frac{1}{2}m\omega^2 x^2$. They also repel one another with a force arising from an interaction potential $V_{\text{int}}(x_1, x_2) = -\frac{1}{4}\kappa m\omega^2(x_1 - x_2)^2$ with $\kappa < 1$ is a real constant.

   (a) Define new coordinates $y_\pm = (x_1 \pm x_2)/\sqrt{2}$. Show that if the particles are bosons, then the wave function when written as a function of $y_\pm$ must be an even function of $y_-$. What if the particles are fermions?

   (b) Show that the time-independent Schrödinger equation separates when written in terms of $y_\pm$.

   (c) Given that the normalised ground state and first excited state wave functions of the standard harmonic oscillator are given by

   $$\psi_0(x) = (\pi\sigma^2)^{-\frac{1}{4}} \exp\left(-\frac{x^2}{2\sigma^2}\right) , \quad \psi_1(x) = \frac{\sqrt{2}x}{\sigma}\psi_0(x) , \quad \text{where} \quad \sigma^2 = \frac{\hbar}{m\omega} ,$$

   determine the ground state wave function of the coupled system when the particles are bosons and when they are fermions.

   Determine energy spectrum in both cases.

   (d) Show that the expected distance $|x_1 - x_2|$ between the particles in the ground state is twice as large for fermions as for bosons. This is an example of what is sometimes called *Pauli repulsion* or *degeneracy pressure*.