B7.3 Further Quantum Theory

Sheet 4 — HT21

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4.1 Anharmonic oscillator WKB

Consider a (possibly anharmonic) oscillator in one dimension, with Hamiltonian

$$H = \tfrac{1}{2m}P^2 + \left(\tfrac{1}{2}m\omega^2 X^2\right)^k \ ,$$

for k a positive integer.

1. Derive the (integral) Bohr-Sommerfeld quantisation condition that implicitly determines the quantum energy levels of this system in the semi-classical approximation.

2. By rescaling the integrand, show that the energy levels are given by

$$E_n = \left(\frac{\pi^{1/2}\Gamma\left(\frac{3k+1}{2k}\right)}{2\Gamma\left(\frac{2k+1}{2k}\right)}\hbar\omega(n+\tfrac{1}{2})\right)^{\frac{2k}{k+1}} \ ,$$

   where you will need the integral identity

$$\int_{-1}^{1}\sqrt{1-x^{2k}}\,\mathrm{d}x = \frac{\pi^{1/2}\Gamma\left(\frac{2k+1}{2k}\right)}{\Gamma\left(\frac{3k+1}{2k}\right)} \ .$$

3. Check your results for the case $k = 1$, which corresponds to the simple harmonic oscillator.

4.2 Spherical WKB approximation

1. Find the WKB approximate wave functions for stationary, s-orbital, bound states (*i.e.*, states with angular momentum $l = 0$ and $E < 0$) of an electron of mass m in a Coulomb potential, with the usual Hydrogen-like Hamiltonian,

$$H = \frac{P^2}{2m} - \frac{Z{q_e}^2}{r} \ .$$

2. Imposing that the wave function is bounded at $r = 0$ leads to a Bohr-Sommerfeld-like quantisation condition. Solve this condition and compare your results with the exact answer for the spectrum of the Hydrogen atom.

3. Now analyse the case with angular momentum $\ell \neq 0$. Here there is both an inner and an outer turning point and you should use the appropriate Bohr-Sommerfeld condition.

   Compare your answer to the exact energy levels. For this purpose, you should recall that the principle quantum number in the Hydrogen atom is given by $n = k + \ell + 1$, where k is the number of zeroes of the radial wavefunction.

4. The *Langer correction* to the WKB analysis of the Hydrogen atom proceeds by replacing $\ell(\ell+1)$ in the centrifugal potential term with $(\ell + \frac{1}{2})^2$. This includes the case $\ell = 0$. Check that the WKB predictions for the energy levels improves dramatically upon implementing the Langer correction.

You will probably need the following integral identity

$$\int_a^b \frac{\sqrt{(r-a)(b-r)}}{r}\,\mathrm{d}r = \frac{\pi}{2}\left(\sqrt{a} - \sqrt{b}\right)^2 , \qquad a < b .$$

4.3 Quantum tunnelling with WKB

A particle is incident from the left upon a potential barrier of the form

$$V(x) = \begin{cases} 0 , & x < -a , \\ a^2 - x^2 , & -a < x < a , \\ 0 , & a < x . \end{cases}$$

1. Set up approximate WKB wave functions for the (non-normalisable) stationary states for this system for both cases $E > a^2$ and $E < a^2$.

   For the latter case, determine the *transmission amplitude* through, and the *reflection amplitude* off of, the classical barrier as a function of the energy of the incident particle.