ELECTROMAGNETISM (PART B)

Chapter 1: Electrostatics (A)

Lecture 2

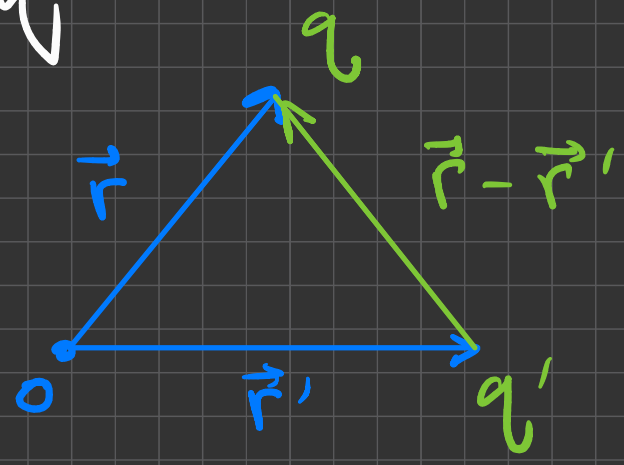
1 Electrostatics

electric phenomena involving time independent distributions of charges and fields

1.1 Coulomb's Law

Coulomb in a series of experiments was able to determine the force between small charged bodies at rest with respect to each other.

Consider two point charges q & q' at positions $\vec{r}$ and $\vec{r}\,'$ with respect to a coordinate system.



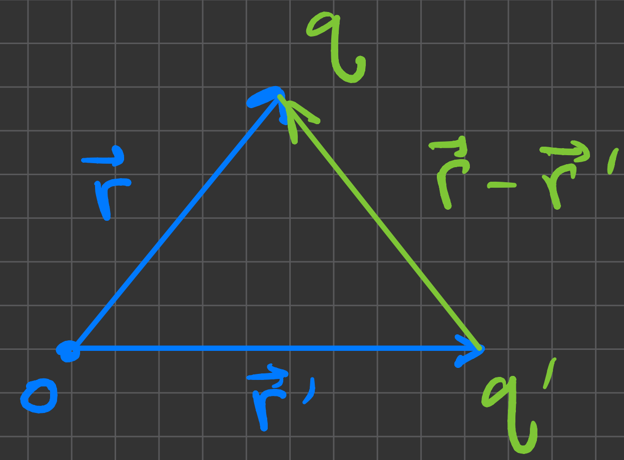
The electric force on the charge q due to the charge q' is given by

$$\boxed{\vec{F}(\vec{r}) = \underbrace{\frac{1}{4\pi\epsilon_0}} \, q q' \, \frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3}}$$

Coulomb's constant $= 8.9879 \times 10^{9} \dfrac{\text{Nt}\cdot\text{mt}^2}{(\text{Coul})^2}$

ϵ_0 : permittivity of free space

$$\boxed{\vec{F}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\, q q' \,\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}}$$



Remarks:

- $|\vec{F}|$ is inversely proportional to the square of the distance between the particles
- $\vec{F}$ is a <u>central force</u> (it depends only on the distance between the charges and it is directed along the line joining the particles)
- $|\vec{F}|$ is proportional to the product of the charges

  $qq' < 0 \rightarrow$ attractive force (opposite charges attract)

  $qq' > 0 \rightarrow$ repulsive force (like charges repel)

Example: Hydrogen atom

compare the gravitational & electrostatic force between the electron e^- and the proton p

$m_e \sim 9 \times 10^{-31}$ kg $\qquad m_p \sim 1.7 \times 10^{-27}$ kg

$-q_e = q_p = 1.6 \times 10^{-19}$ Coul

on average: e^- & p are separated by a distance of $\sim 5 \times 10^{-11}$ mt

Then:

$$\left.\begin{aligned} F_{grav} &\sim 3.6 \times 10^{-47} \text{ Nt} \\ F_{coul} &\sim 8 \times 10^{-8} \text{ Nt} \end{aligned}\right\} \quad \frac{F_{coul}}{F_{grav}} \sim 2 \times 10^{39}!$$

The electric field

Although what gets measured in practice is the force $\vec{F}$, we think instead in terms of an electric field $\vec{E}$ due to a given distribution of charge.

A point charge q' at a position $\vec{r}'$ generates an electric field $\vec{E}(\vec{r})$ at all other points $\vec{r}$ in space given by

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} q' \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$

Then, a (test) particle with charge q at $\vec{r}$ "feels" a force $\boxed{\vec{F} = q\vec{E}}$ when placed in the field $\vec{E}$ generated by q'.

Using the superposition principle, we can generalise Coulomb's law to more general distributions of charges.

The electric field $\vec{E}(\vec{r})$ at a point $\vec{r}$ due to a system of point charges $q_1, \ldots, q_n$ located at $\vec{r}_1, \ldots, \vec{r}_n$ is the vector sum

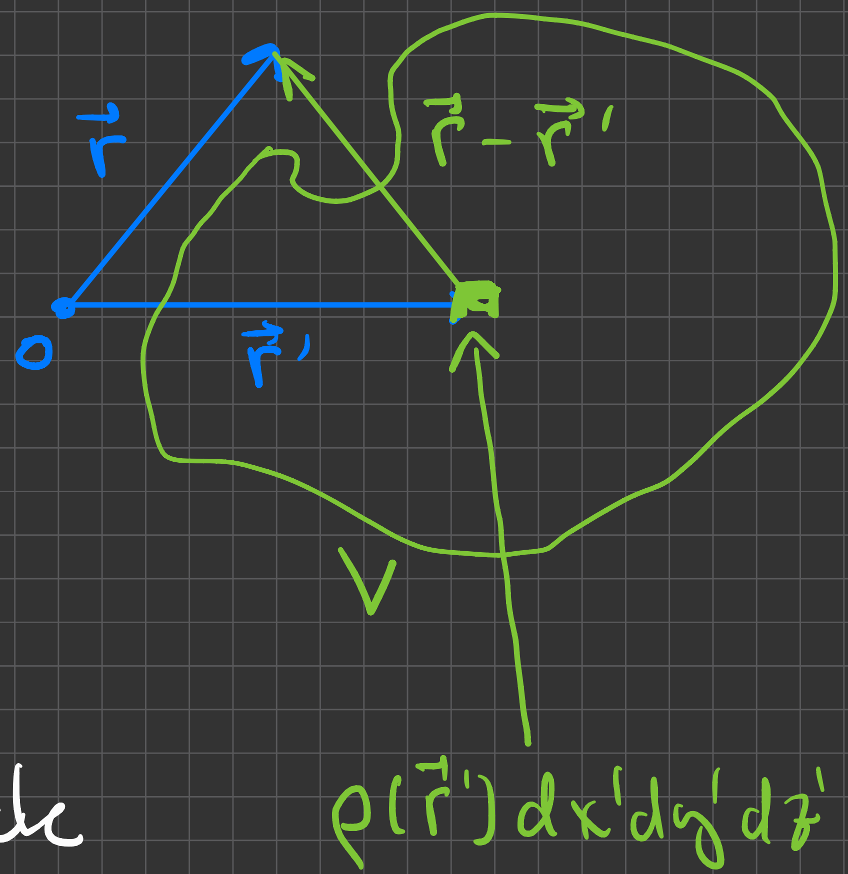
$$\vec{E}(\vec{r}) = \sum_{i=1}^{n} \vec{E}_i(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} q_i \frac{\vec{r}-\vec{r}_i}{|\vec{r}-\vec{r}_i|^3}$$

Then a test particle with charge q at $\vec{r}$ is subject to a force

$$\vec{F}(\vec{r}) = q\,\vec{E}(\vec{r})$$

We can generalise this further a <u>volume distribution</u> of a continuous charge density $\rho(\vec{r})$ in a region V. We have:

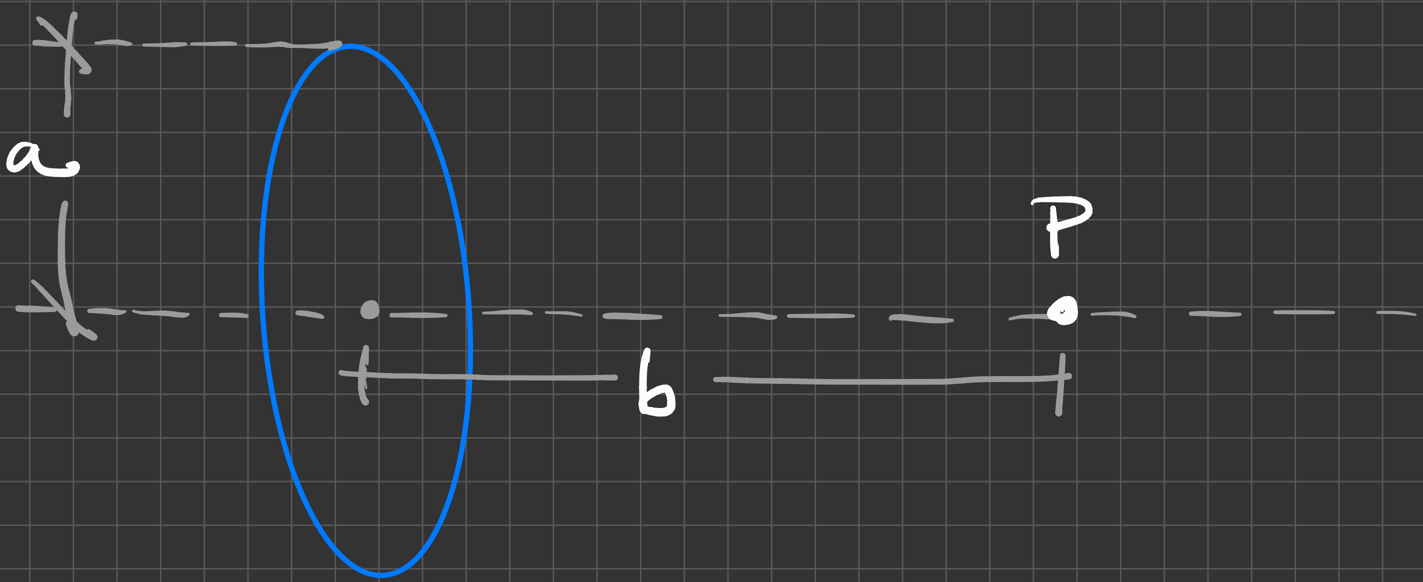
$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\rho(r')\,dx'dy'dz'$$



is the electric field due to $\rho(\vec{r})$

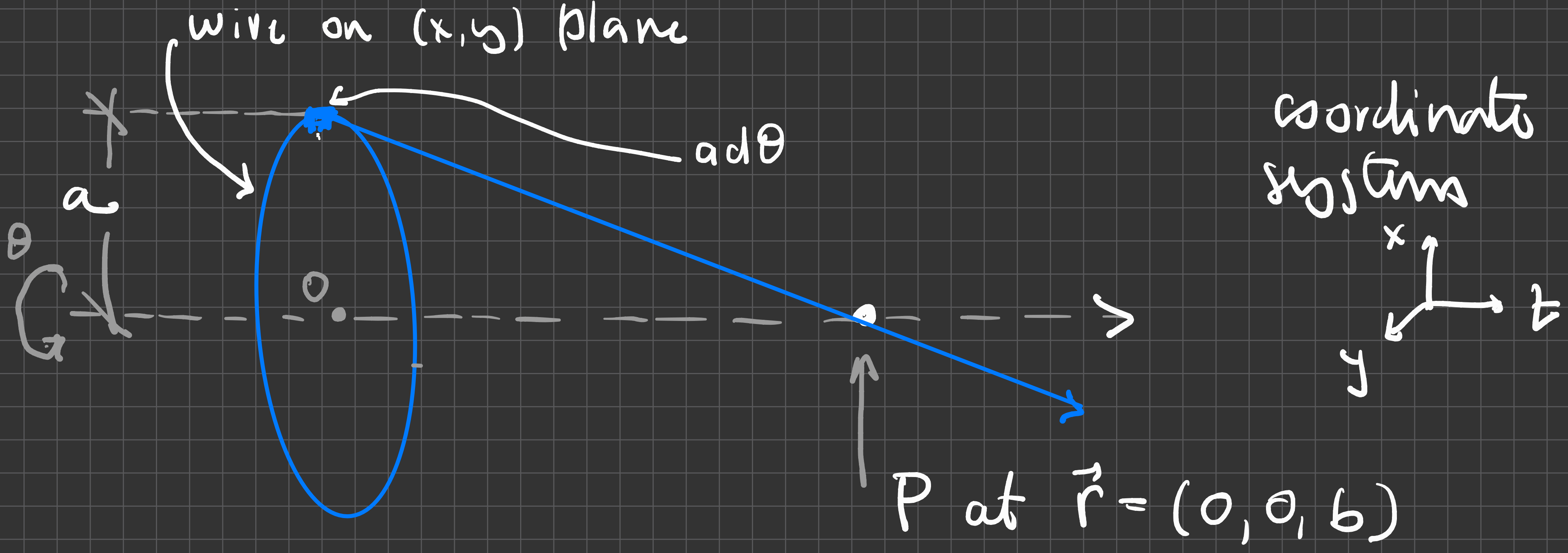
Again: $\vec{F}(\vec{r}) = q\,\vec{E}(\vec{r})$
is the force on a test particle with charge q at $\vec{r}$.

Example Consider a plane circular wire of radius a and suppose it has a total charge Q uniformly distributed around the wire.



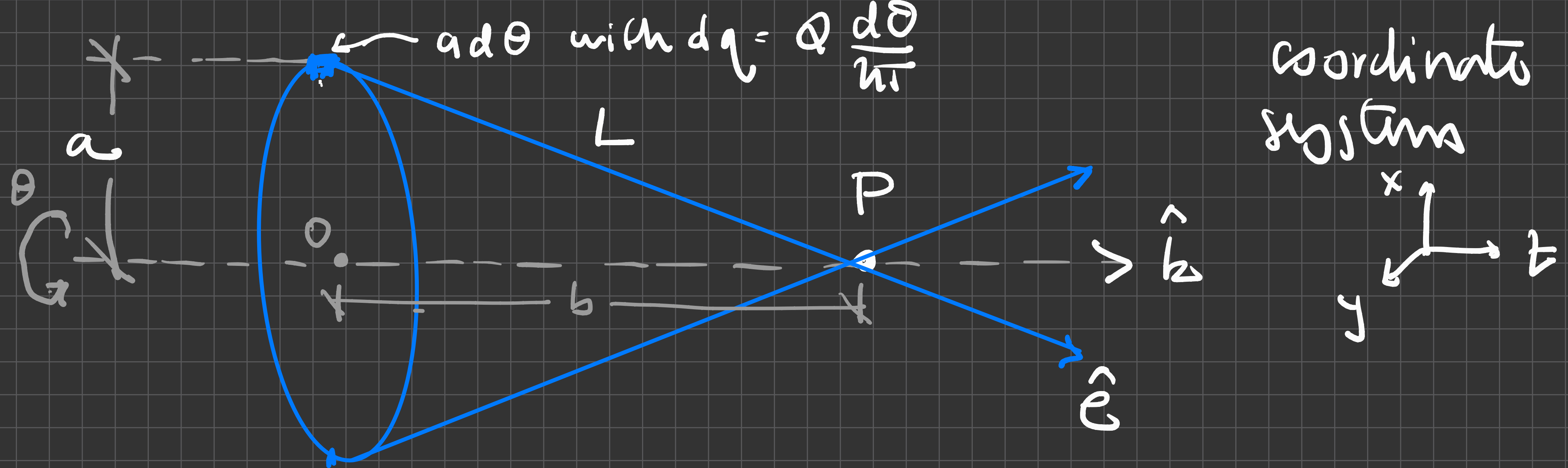
Find $\vec{E}$ at a point P on the axis of the circle at a distance b from the centre

We will do this directly from Coulomb's law by cutting the wire into small elements and then use the superposition principle to add up all the contributions.



cut wire into segments each of length $a d\theta$

each segment contains a charge $\quad dq = Q \frac{d\theta}{2\pi}$



contribution to $\vec{E}$ at P of dq

$$= \frac{1}{4\pi\epsilon_0}\,\frac{Q\,d\theta}{2\pi}\,\frac{1}{L^3}\,(L\hat{e}) \quad , \quad L^2 = a^2 + b^2$$

Note that the components perpendicular to the z-axis of diametrically opposite wire segments cancel each other!

Then, adding all contributions around the circle

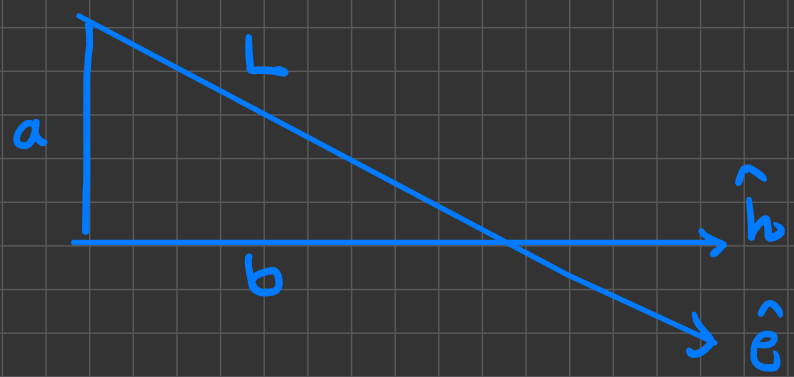
$$\vec{E} = E\,\hat{z} \qquad \text{by symmetry!}$$

ie $\vec{E}$ is along z-axis.

So, we only need to compute the z-component of $\vec{E}$

$$E = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} \frac{Q}{2\pi L^2}\,\underbrace{\hat{e}\cdot\hat{z}}\,d\theta = \frac{1}{4\pi\epsilon_0}\,Q\,\frac{b}{(a^2+b^2)^3}$$

$$\hat{e}\cdot\hat{z} = \frac{b}{L}$$



1.2 The electrostatic scalar potential

Recall the formula for the electrostatic field $\vec{E}(\vec{r})$ due to a discrete distribution of point charges

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n} q_i \frac{\vec{r}-\vec{r}_i}{|\vec{r}-\vec{r}_i|^3}$$

Note that

$$\vec{E}(\vec{r}) = -\nabla\Phi(\vec{r})$$

where

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{|\vec{r}-\vec{r}_i|}, \quad \forall\, \vec{r}\neq\vec{r}_i$$

ie $\vec{E}(\vec{r})$ is the gradient of a function & we call $\Phi(\vec{r})$ the electric or scalar potential.

$$\vec{E}(\vec{r}) = -\nabla\,\Phi(\vec{r})$$

The potential is defined up to a constant: we can add a constant to Φ without changing $\vec{E}$

Note moreover: $\boxed{\nabla \wedge \vec{E} = 0}$

This is one of Maxwell's equations for time independent distributions of charges.

Given $\vec{E}$ st $\nabla \wedge \vec{E} = 0$ then one can conclude that $\vec{E} = -\nabla\Phi$ as long as the region of space $V \subseteq \mathbb{R}^3$ being considered is simply connected.

(We do not need to worry about this for the time being)

Physical interpretation:

Consider a point particle with charge q placed in the electric field $\vec{E}$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n} q_i \frac{\vec{r}-\vec{r}_i}{|\vec{r}-\vec{r}_i|^3}$$

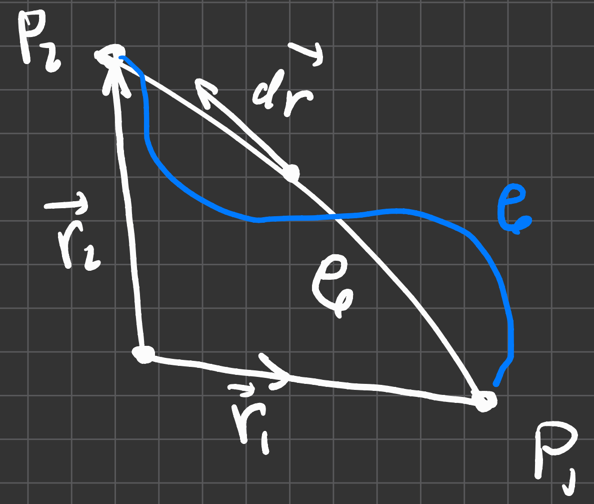
generated by a discrete distribution of charges.

The particle then experiences a force

$$\vec{F}(\vec{r}) = q\,\vec{E}(\vec{r}) = -q\,\nabla\Phi$$

so $\vec{F}$ is a conservative force.

Consider the work done against the electrostatic force in moving the charge q along a path $\mathcal{C}$ from the point P_1 at position $\vec{r}_1$ to the point P_2 at $\vec{r}_2$:



$$W = -\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = -q \int_{\mathcal{C}} \vec{E} \cdot d\vec{r}$$

$$= q \int_{\mathcal{C}} \nabla \Phi \cdot d\vec{r}$$

$$W = q \left(\Phi(\vec{r}_2) - \Phi(\vec{r}_1) \right)$$

This is independent of the path (as the force is conservative)

$$W = q\left(\Phi(\vec{r}_2) - \Phi(\vec{r}_1)\right)$$

work done in moving charge from P_1 to P_2 $=$ potential energy per unit of charge

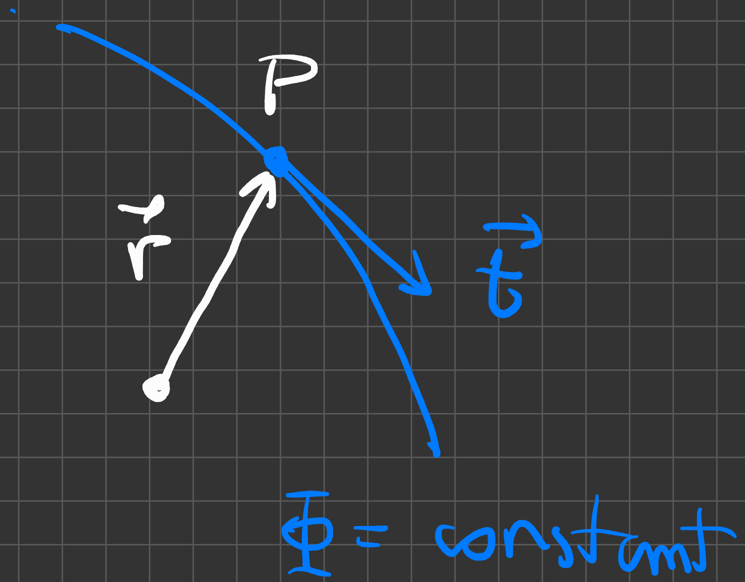
Recall that the potential is defined up to a constant: it is only the difference in the values of Φ that are physical.

(This difference is called voltage.)

Surfaces of constant Φ are called <u>equipotentials</u>.

The electric field is always normal to equipotentials

Consider a vector $\vec{t}$ at the point P with position $\vec{r}$, which is tangent to an equipotential surface for $\Phi(\vec{r})$. Then



$$\vec{t} \cdot \nabla \Phi = 0 \quad \text{at } P$$

$\Longrightarrow \quad \vec{E} = -\nabla \Phi$ is normal to the equipotential

Conservation of energy:

Consider Newton's equations of motion for a particle with mass m & charge q placed in an electrostatic field $\vec{E}(\vec{r})$

The particle then experiences a force

$$\vec{F}(\vec{r}) = q\,\vec{E}(\vec{r}) = -q\,\nabla\Phi = m\vec{a}$$

The kinetic energy of the particle is

$$T = \frac{1}{2} m\,\vec{v}\cdot\vec{v}\,, \qquad \vec{v} = \dot{\vec{r}} \text{ (velocity)}$$

Then

$$\frac{d}{dt} T = \frac{1}{2} m \frac{d}{dt} (\vec{v} \cdot \vec{v}) = m \vec{v} \cdot \vec{a} = \vec{v} \cdot \vec{F}$$

$$= -q \vec{v} \cdot \nabla \Phi = -q \sum_{i=1}^{3} \frac{dx^i}{dt} \frac{\partial \Phi}{\partial x^i} = -q \frac{d\Phi}{dt}$$

ie :

$$\frac{d}{dt} (T + q\Phi) = 0$$

Hence $\boxed{\mathcal{E} = T + \underline{q \Phi}}$ is a constant in time

potential energy of the particle in the electric field $\vec{E}$

1.3 The Dirac delta function

Recall

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} \rho(\vec{r}\,')\, dx'dy'dz'$$

is the electrostatic field for a continuous distribution of charge with charge density $\rho(\vec{r})$

Question: Can one reduce this expression to that corresponding to a discrete distribution of charge?

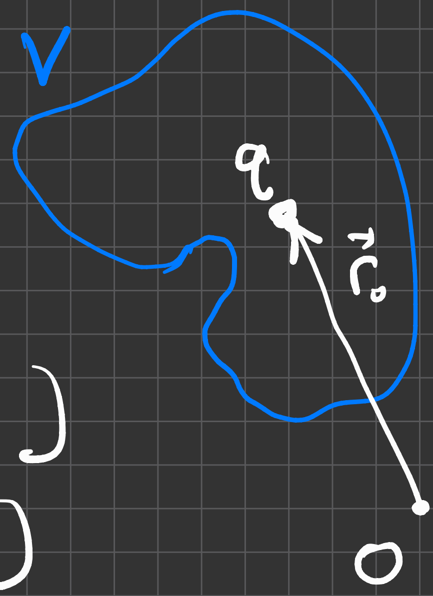
In order to answer this question we need to understand which charge density $\rho(\vec{r})$ corresponds to a point charge with charge q at a point $\vec{r}_0$.

This function $\rho(\vec{r})$ should be such that

- it <u>vanishes</u> "outside" the point $\vec{r}_0$, $\forall t$

<u>and</u>

- $$\int_V \rho(\vec{r})\, dx\, dy\, dz = q$$



$\forall \; V \subset \mathbb{R}^3$ (regions V in space $\mathbb{R}^3$)
which <u>contain</u> $\vec{r}_0$ $\quad (\vec{r}_0 \in V)$

Define: the one-dimensional Dirac delta function $\boxed{\delta(x-a)}$ by the following properties

(1) $\quad \delta(x-a) = 0 \quad \forall\ x \neq a$

(2) $$\int_{R \subset \mathbb{R}} \delta(x-a)\,dx = \begin{cases} 1 & \text{if } a \in R \\ 0 & \text{otherwise} \end{cases}$$

Remark: mathematically this is an example of an improper function (distribution).
For this lecture course we only need an intuitive notion (see Jackson: We can think of the δ-function as "the limit of a peaked curve as it becomes narrower & narrower but higher & higher st the area under the curve is always constant.")

From the definition, one can derive useful properties

(3) $$\int_R f(x)\,\delta(x-a)\,dx = f(a) \quad \text{if} \quad a \in R \subset \mathbb{R}$$

evaluates f at a!

(4) $$\delta(f(x)) = \sum_{i=1}^{N} \frac{1}{|f'(x_i)|}\,\delta(x-x_i)$$

where $f(x_i)=0$ are simple zeros at $x=x_i$.
(f is a continuously differentiable function)

Note then that $\delta(f(x))=0$ if f is nowhere zero.

In higher dimensions $\delta(\vec{r}-\vec{r}')$ is just the product of the Cartesian δ-functions

(5) $$\delta(\vec{r}-\vec{r}') = \delta(x-x')\,\delta(y-y')\,\delta(z-z')$$

vanishes everywhere except at $\vec{r}=\vec{r}'$

(6) $$\int_V \delta(\vec{r}-\vec{r}')\,dx'dy'dz' = \begin{cases} 1 & \text{if } \vec{r}\in V\subset\mathbb{R}^3 \\ 0 & \text{otherwise} \end{cases}$$

[As we will see later: the δ-function will help to make sense of the differentiation of "functions" whose derivatives do not exist eg

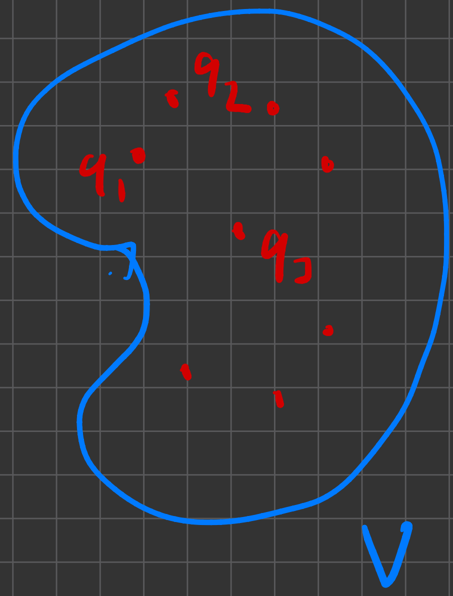
$$\frac{d}{dr}\left(\frac{1}{r}\right)$$
]

Return to our question: which charge density $\rho(\vec{r})$ corresponds to a discrete distribution of charges $q_1, \ldots, q_n$ at $\vec{r}_1, \ldots, \vec{r}_i$?

Answer:
$$\rho(\vec{r}) = \sum_{i=1}^{n} q_i \, \delta(\vec{r} - \vec{r}_i)$$

- it vanishes "outside" the points $\vec{r}_i$, $\forall t$
- and

$$\int_V \rho(\vec{r}) \, dx\,dy\,dz$$
$$= \sum_{i=1}^{n} q_i \int_V \delta(\vec{r} - \vec{r}_i) \, dx\,dy\,dz$$
$$= \sum_{i=1}^{n} q_i = Q \text{ - total charge in } V$$



Moreover: recall

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} \rho(\vec{r}')\, dx'dy'dz'$$

is the electrostatic field for a continuous distribution of charge with charge density $\rho(\vec{r})$

Substituting $\rho(\vec{r}) = \sum_{i=1}^{n} q_i\, \delta(\vec{r}-\vec{r}_i)$ into $\vec{E}$:

$$\vec{E}(\vec{r}) = \sum_{i=1}^{n} \frac{q_i}{4\pi\epsilon_0} \int_V \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} \delta(\vec{r}'-\vec{r}_i)\, dx'\, dy'\, dz'$$

property (3)

$$= \sum_{i=1}^{n} \frac{q_i}{4\pi\epsilon_0} \frac{\vec{r}-\vec{r}_i}{|\vec{r}-\vec{r}_i|^3}$$

So for $\rho(\vec{r}) = \sum_{i=1}^{n} q_i\, \delta(\vec{r}-\vec{r}_i)$, $\vec{E}(\vec{r})$ above reduces to the electric field of a discrete collection of charges.

1.4 Gauss' law and Poisson's equation

So far we have seen that for a collection of n point charges with charge $q_1, \dots, q_n$ at $\vec{r}_i$ we have

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} q_i \frac{\vec{r}-\vec{r}_i}{|\vec{r}-\vec{r}_i|^3}$$

We can check explicitly that

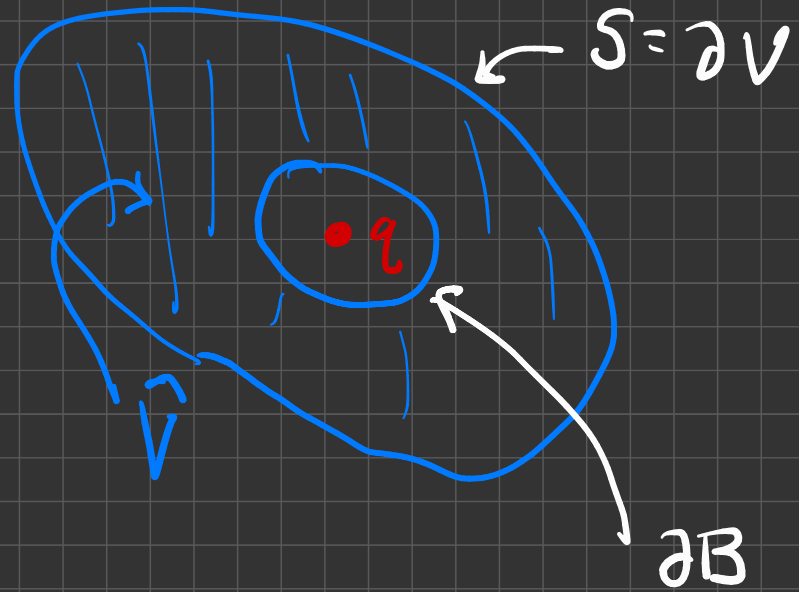
$$\nabla \cdot \vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} q_i \left(\frac{3}{|\vec{r}-\vec{r}_i|^3} - \frac{3(\vec{r}-\vec{r}_i)\cdot(\vec{r}-\vec{r}_i)}{|\vec{r}-\vec{r}_i|^5} \right)$$

$$= 0 \qquad \forall \quad \vec{r} \neq \vec{r}_i$$

Consider first the case of <u>one</u> particle at the origin $\vec{r}=0$ with charge q

Let S be a surface bounding a region V $(S=\partial V)$ which contains the charge

Let ∂B be a small sphere centered at the origin (thus containing the charge) bounding a ball B



Then on $\hat{V} = V \setminus B$

$$\nabla \cdot \vec{E} = 0 \qquad \text{as} \qquad \vec{r}=0 \notin \hat{V}$$

Hence:

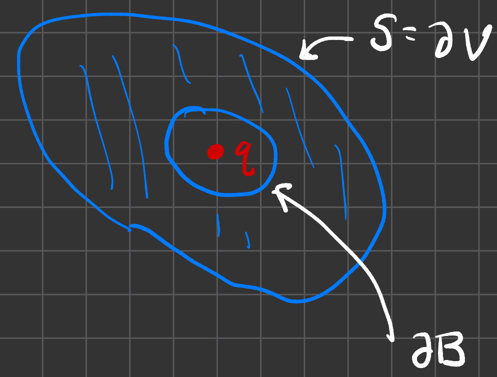
$$0 = \int_{\hat{V}} \nabla \cdot \vec{E}\, dV = \int_{\partial \hat{V}} \vec{E} \cdot d\vec{S}$$

by the divergence theorem

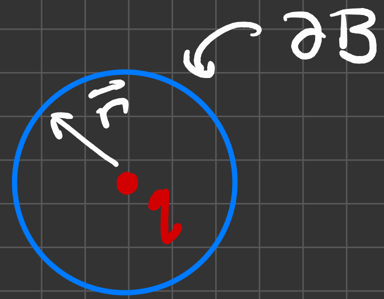
$d\vec{S} = \hat{n}\, dS$

$\hat{n}$ unit outward normal vector to $\hat{S} = \partial \hat{V}$

$$= \int_{S = \partial V} \vec{E} \cdot d\vec{S} - \int_{\partial B} \vec{E} \cdot d\bar{S}$$



$\Longrightarrow$



$$\int_S \vec{E}\cdot d\vec{S} = \int_{\partial B} \vec{E}\cdot d\vec{S}$$

sphere of radius r

electric field due to the charge q at $\vec{r}=0$

$$= \int_{\partial B} \frac{q}{4\pi\epsilon_0}\,\frac{\vec{r}\cdot\hat{n}}{r^3}\,dS \;,$$

$\hat{n} = \dfrac{\vec{r}}{r}$ unit normal to ∂B

$dS = r^2 \sin\theta\, d\theta\, d\phi$

area element in spherical coordinates

$0 \le \theta < \pi, \quad 0 \le \phi < 2\pi$

$$= \int_{\partial B} \frac{q}{4\pi\epsilon_0}\,\frac{1}{r^2}\, r^2 \sin\theta\, d\theta\, d\phi = \frac{q}{4\pi\epsilon_0}\cdot 4\pi = \frac{q}{\epsilon_0}$$

so

$$\int_S \vec{E} \cdot d\vec{S} = \begin{cases} \dfrac{q}{\epsilon_0} & \text{if the charge is contained in } V \text{ bounded by } S = \partial V \\ 0 & \text{otherwise} \end{cases}$$

We can now extend this result to a collection of n point particles with charges $q_1, \dots, q_n$ at $\vec{r}_1, \dots, \vec{r}_n$ contained in a region $V \subset \mathbb{R}^3$ bounded by a surface $S = \partial V$.

For each charge, let

$S_i = \partial B_i$ be a small sphere bounding a ball B_i which contains the charge q_i

Consider the region

$$\hat{V} = V \setminus \sum B_i$$

On $\hat{V} = V \setminus \sum_{i=1}^{n} B_i$ we have

$$\nabla \cdot \vec{E} = 0 \qquad \forall\ \vec{r} \neq \vec{r}_i$$

By the divergence theorem we have

$$0 = \int_{\hat{V}} \nabla \cdot \vec{E}\ dV = \int_{\partial \hat{V}} \underline{E} \cdot d\vec{S}$$

$$= \int_{S} \vec{E} \cdot d\vec{S} - \sum_{i=1}^{n} \int_{S_i} \vec{E} \cdot d\vec{S}$$

electric field due to q_i at $\vec{r}_i$ (the other charges q_j, $j \neq i$, do not contribute as they are outside the region bounded by S_i)

$$\Rightarrow \int_S \vec{E} \cdot d\vec{S} = \sum_{i=1}^{n} \int_{S_i} \vec{E} \cdot d\vec{S} = \frac{1}{\epsilon_0} \sum_{i=1}^{n} q_i$$

We then obtain

$$\therefore \quad \int_S \vec{E} \cdot d\vec{S} = \frac{1}{\epsilon_0} Q$$

$Q = \sum_{i=1}^{n} q_i =$ total charge enclosed by S

We <u>assume</u> (by the superposition principle) that this is also true for a continuous charge density $\rho(\vec{r})$

$$\boxed{\begin{aligned} &\int_S \vec{E}\cdot d\vec{S} = \frac{1}{\epsilon_0} Q = \frac{1}{\epsilon_0}\int_V \rho(\vec{r})\, dV \\ &Q = \int_V \rho\, dV = \text{total charge enclosed by } S = \partial V \end{aligned}}$$

(Integral form of) Gauss' law

$\rightsquigarrow$ flux of $\vec{E}$ out of $V = \frac{1}{\epsilon_0}$ (total charge inside V)

Differential form of Gauss' law:

By the divergence theorem on the LHS $(S=\partial V)$

$$\int_S \vec{E}\cdot d\vec{S} = \int_V \nabla\cdot\vec{E}\, dV$$

$$\Rightarrow \quad \int_V \left(\nabla\cdot\vec{E} - \frac{1}{\epsilon_0}\rho(\vec{r})\right) dV = 0$$

As this must be true for all possible regions V we obtain

Maxwell Eq #1

$$\boxed{\nabla\cdot\vec{E} = \frac{1}{\epsilon_0}\rho(\vec{r})}$$

differential version of Gauss' law

(true even if ρ depends on t: chapter 4)

Remark: For a particle with charge q at $\vec{r}_0$: $\rho(\vec{r}) = q\,\delta(\vec{r}-\vec{r}_0)$

$$\Rightarrow \quad \nabla \cdot \vec{E} = \frac{1}{\epsilon_0} q\,\delta(\vec{r}-\vec{r}_0)$$

which expresses the fact that

$$\nabla \cdot \vec{E} = 0 \quad \text{when} \quad \vec{r} \neq \vec{r}_0$$

and

$$\int_S \vec{E} \cdot d\vec{S} = \frac{1}{\epsilon_0} q$$

(flux out of S enclosing the charge $= \frac{1}{\epsilon_0} q$)

Given a configuration of charges Gauss' law is not enough to determine $\vec{E}$, because Gauss' law is one scalar equation for 3 components of $\vec{E}$.

However a vector field is "completely" determined if its divergence and its curl are given for all $\vec{r} \in \mathbb{R}^3$ (up to a ∇f st $\nabla^2 f = 0$, f a function) Helmoltz theorem

We have shown that for a discrete distribution of point charges at $\vec{r}_i$ $\nabla \wedge E = 0$ for $\vec{r} \neq \vec{r}_i$

It is not too hard to show that this is true for a continuous distribution of charge with charge density $\rho(\vec{r})$ in a region $V \subset \mathbb{R}^3$.

Note that in this case we have

$$\vec{E}(\vec{r}) = -\nabla \Phi(\vec{r})$$

where

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\vec{r}\,') \frac{1}{|\vec{r}-\vec{r}\,'|} \, dV'$$

because

$$\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} = -\nabla\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right) \qquad \vec{r} \neq \vec{r}\,'$$

If $\rho(\vec{r})$ is differentiable, then so is $\vec{E}$.

Hence $\boxed{\nabla \wedge \vec{E}(\vec{r}) = 0}$ MEq #2 for electrostatics

Theorem: (Calculus in 3-dims, Prelims)

Let $f(\vec{r})$ be a bounded continuous function with support $\{\vec{r} \in \mathbb{R}^3 \mid f(\vec{r}) \neq 0\} \subset V$ in a bounded region $V \subset \mathbb{R}^3$.

Let
$$F(\vec{r}) = \int_V \frac{f(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}\, dV'$$

Then $F(\vec{r})$ is differentiable in $\mathbb{R}^3$ with

$$\nabla F(\vec{r}) = -\int_V f(\vec{r}\,') \frac{(\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3}\, dV'$$

Both F & ∇F are continuous and tend to zero as $r \to \infty$. Moreover, if f is differentiable, then ∇F is differentiable and $\nabla^2 F = -4\pi f$

$\boxed{\text{Summarying:}}$

$$\nabla \cdot \vec{E} = \frac{1}{\epsilon_0} \rho \qquad \text{Gauss}$$

$$\nabla \wedge \vec{E} = 0 \iff \vec{E} = -\nabla \Phi$$

(simply connected region)

Combining these into a single eq:

$$\boxed{\nabla^2 \Phi = -\frac{1}{\epsilon_0} \rho}$$

Poisson's eq for Φ

Any relation gives $\vec{E}$ by $\vec{E} = -\nabla \Phi$

Remark: we expect that the expression

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\vec{r}') \frac{1}{|\vec{r}-\vec{r}'|} \, dV'$$

for the electrostatic potential due to a charge density $\rho(\vec{r})$ satisfies Poisson's eq.

This is in fact true:

$$\nabla^2 \Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\vec{r}') \, \nabla^2\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) dV'$$

because $\boxed{\nabla^2\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) = -4\pi\,\delta(\vec{r}-\vec{r}')}$.

So

$$\nabla^2 \Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}(-4\pi)\int_V \rho(\vec{r}')\,\delta(\vec{r}-\vec{r}')\,dV' = -\frac{1}{\epsilon_0}\rho(\vec{r}) \checkmark$$

$$\nabla^2 \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) = -4\pi\, \delta(\vec{r} - \vec{r}') \quad \text{precisely captures}$$

(a) $\nabla^2 \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) = 0 \quad \forall\ \vec{r} \neq \vec{r}'$ (can easily prove this)

(b) On the other hand integrating on both sides

$$\int_{\vec{r} \in V} \nabla^2 \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) dV' = -4\pi$$

It requires a little work to prove this as we need to be careful where $\vec{r} = \vec{r}'$

Noting first that

$$\int_{\vec{r}\in V} \nabla^2\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right) dV' = + \int_{\vec{r}\in V} \nabla'^2 \left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right) dV'$$

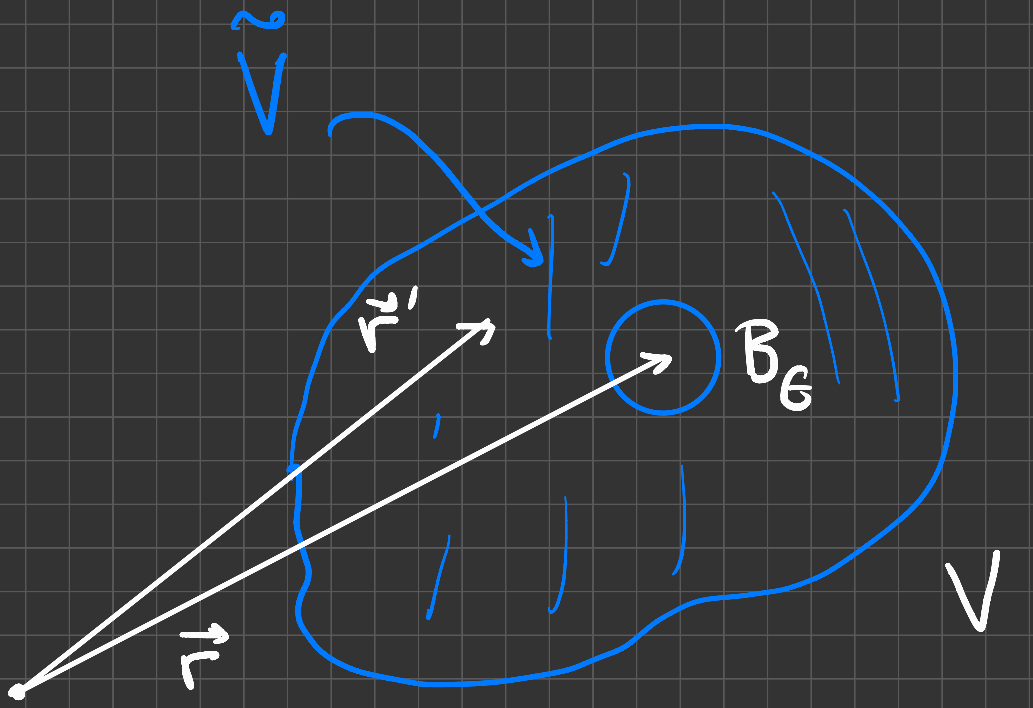
we simplify our computations WLOG by translating the origin to $\vec{r}$. We then integrate over a small ball $\underline{0} \in B_\epsilon$

$$\int_{B_\epsilon} \nabla^2\left(\frac{1}{r}\right) dV = \int_{B_\epsilon} \nabla\cdot\nabla\left(\frac{1}{r}\right) dV = \int_{S_\epsilon = \partial B_\epsilon} \nabla\left(\frac{1}{r}\right)\cdot\hat{n}\, dS$$

$$= -\int_{S_\epsilon} \frac{1}{\epsilon^2} \quad \epsilon^2 \sin\theta\, d\theta\, d\phi = -4\pi$$

Consider now

$$\tilde{V} = V \setminus B_\epsilon$$



Then as

$$\int_{\tilde{V}} \nabla^2 \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) dV' = 0 \quad \text{as } \vec{r} \notin \tilde{V}$$

We find

$$\int_{\vec{r} \in V} \nabla^2 \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) dV' = \int_{B_\epsilon} \nabla^2 \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) dV' = -4\pi$$

So indeed.

$$\nabla^2 \left(\frac{1}{|\vec{r} - \vec{r}\,'|} \right) = -4\pi\, \delta(\vec{r} - \vec{r}\,') \qquad \text{precisely captures}$$

(a) $\nabla^2 \left(\frac{1}{|\vec{r} - \vec{r}\,'|} \right) = 0 \quad \forall \; \vec{r} \neq \vec{r}\,'$

(b) $$\int_{\vec{r} \in V} \nabla^2 \left(\frac{1}{|\vec{r} - \vec{r}\,'|} \right) dV' = -4\pi$$

and $$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\vec{r}\,') \frac{1}{|\vec{r} - \vec{r}\,'|} \, dV'$$

satisfies $\nabla^2 \Phi = -\frac{1}{\epsilon_0} \rho(\vec{r})$ ✓ //