ELECTROMAGMETISM (PART B)

Chapter 1: Electrostatics (B)

Lecture 3

Xenia de la Ossa

$\boxed{1}$ Electrostatics

electric phenomena involving time independent distributions of charges and fields

$\boxed{\text{Contents}}$ (so far)

Summarizing: last lecture

$$\nabla \cdot \vec{E} = \frac{1}{\epsilon_0} \rho$$ Gauss' law

$$\nabla \wedge \vec{E} = 0 \iff \vec{E} = -\nabla \Phi$$

Combining these into a single eq

$$\boxed{\nabla^2 \Phi = -\frac{1}{\epsilon_0} \rho}$$ Poisson's eq for Φ

Any relation gives $\vec{E}$ by $\vec{E} = -\nabla \Phi$

Also: $\vec{F} = q\vec{E}$ is the force on a test particle of charge q in the presence of the field $\vec{E}$ generated by $\rho(\vec{r})$

Contents

this lecture

1.5 An important example

illustrate

- Gauss' law
- complications we find when trying to compute $\vec{E}$ or Φ when there are boundaries between regions in space

We have seen that for a continuous distribution of charge $\rho(\vec{r})$ in a bounded region V we have

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\vec{r}' \in V} \rho(\vec{r}') \frac{1}{|\vec{r}-\vec{r}'|} dV'$$

for discrete distribution, $\rho(\vec{r}) = \sum_{i=1}^{N} q_i \delta(\vec{r}-\vec{r}_i)$

solves Poisson eq. $\nabla^2 \Phi = -\frac{1}{\epsilon_0} \rho$

Moreover: $\Phi(\vec{r}) \sim \frac{C}{r}$ as $r \to \infty$

(Can you prove this?)

Of course: $\vec{E}(\vec{r}) = -\nabla \Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\vec{r}' \in V} \rho(\vec{r}') \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} dv'$

BUT: what happens when there are charged surfaces in a region V?

Suppose there is a surface $\Sigma \subset \mathbb{R}^3$ which has a surface charge density σ.

(Σ: eg thin metal surface)

This configuration generates an electric field

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{r' \in \Sigma} \sigma(\vec{r}') \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} \, dS'$$

smooth if σ is smooth

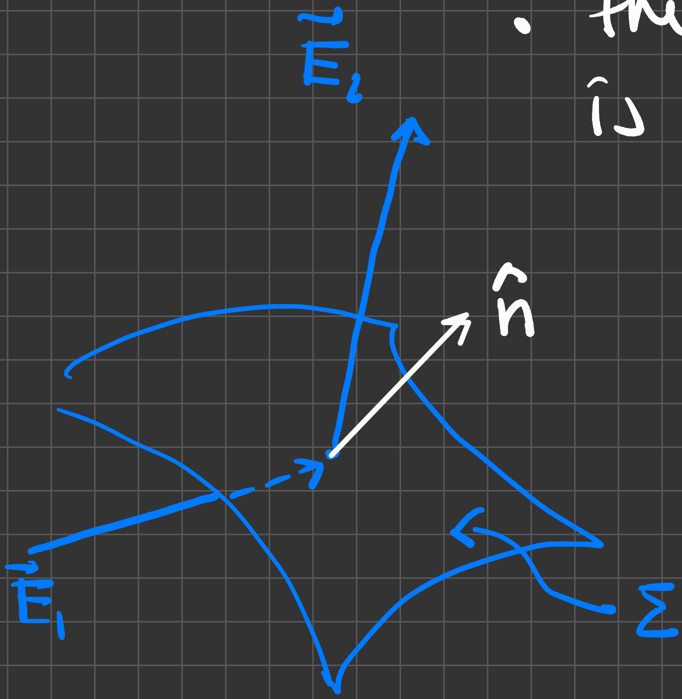
at all points $\underline{\vec{r} \in \mathbb{R}^3 \setminus \Sigma}$.

However: there is discontinuity in $\vec{E}(\vec{r})$ as one crosses Σ !

Claim:

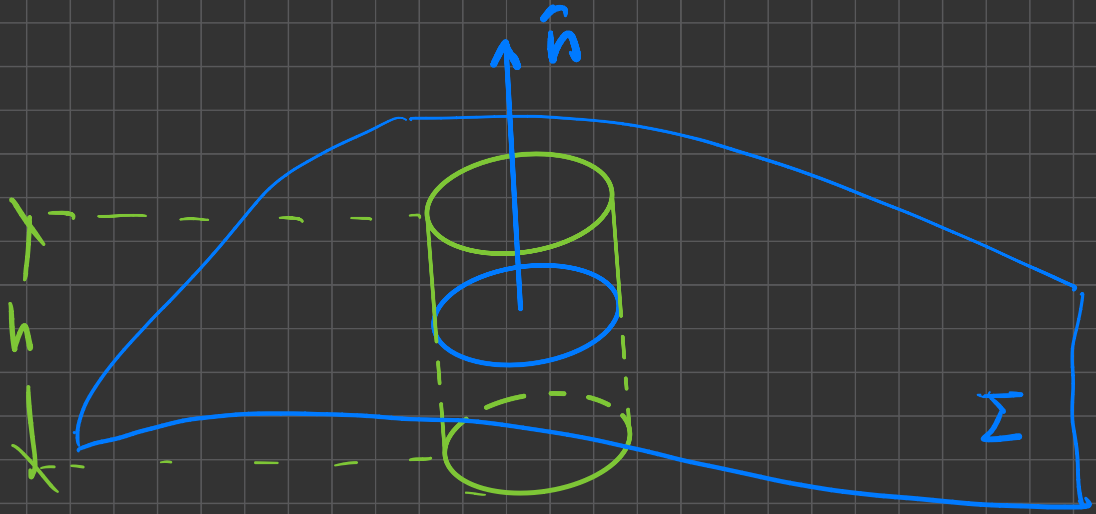
- the components of $\vec{E}$ tangent to Σ are continuous across Σ
- the component of $\vec{E}$ normal to Σ is not continuous and

$$\boxed{\vec{E}_2 \cdot \hat{n} - \vec{E}_1 \cdot \hat{n} = \frac{\sigma}{\epsilon_0}}$$



$\hat{n}$ = normal to Σ in the direction of $\vec{E}$ in region 2

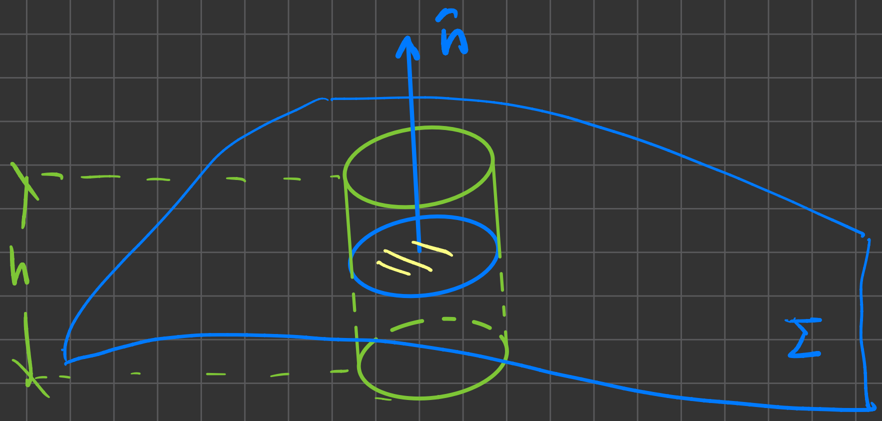
Proof of the claim: use Gauss' law



Consider a region V which is a cylinder of height h and cross sectional area δS

Gauss' law $\Rightarrow$ $$\int_{S=\partial V} \vec{E}\cdot d\vec{S} = \frac{1}{\epsilon_0}\int_S \sigma \, dS$$

$$\int_S \vec{E}\cdot d\vec{S} = \frac{1}{\epsilon_0}\int_S \sigma \, dS$$



$$\int_{S=\partial V} \vec{E}\cdot d\vec{S} = \frac{1}{\epsilon_0}\int_S \sigma \, dS$$

Then in the limit $h \to 0$

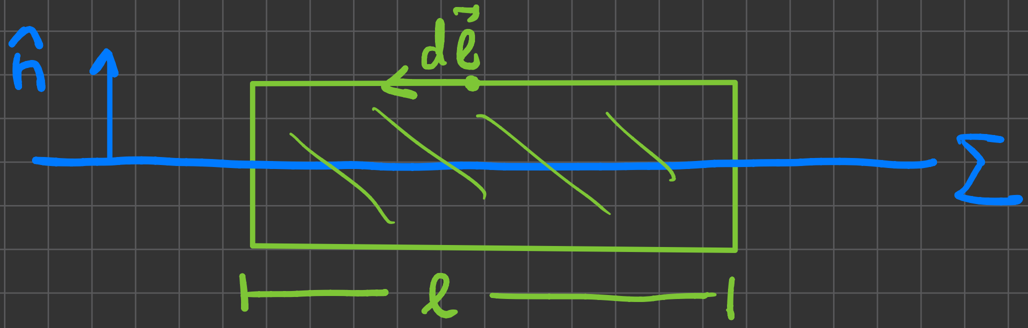
$$\int_S \vec{E}\cdot d\vec{S} = \int_S \vec{E}_2\cdot d\vec{S} - \int_S \vec{E}_1\cdot d\vec{S} = \frac{1}{\epsilon_0}\int_S \sigma \, dS$$

$\hat{n}\, dS$

Then

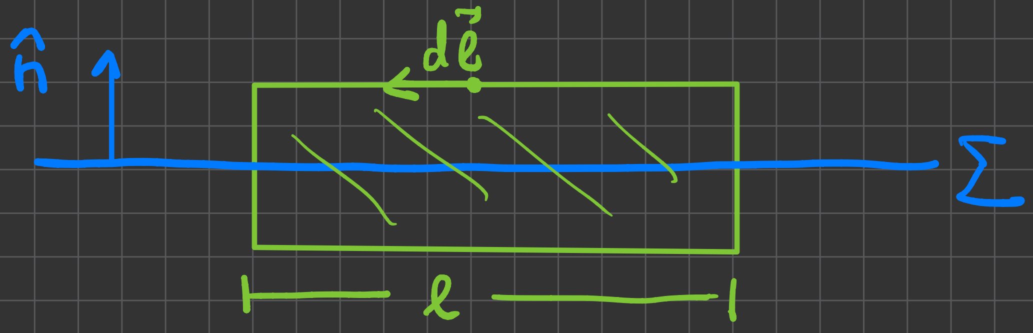
$$\vec{E}_2\cdot\hat{n} - \vec{E}_1\cdot\hat{n} = \frac{1}{\epsilon_0}\sigma$$

consider instead a rectangular loop $\mathcal{C} = \partial A$ bounding a rectangle A of height h and length ℓ



Recall also $\nabla \wedge \vec{E} = 0$. Then

$$0 = \int_A (\nabla \wedge \vec{E}) \cdot \underset{\substack{\uparrow \\ \text{normal} \\ \text{to } \vec{A}}}{\vec{dA}} \underset{\substack{\uparrow \\ \text{Stoke's theorem}}}{=} \int_{\mathcal{C} = \partial A} \vec{E} \cdot d\vec{\ell}$$



Thus $$0 = \int_{\mathcal{C}=\partial A} \vec{E}\cdot d\vec{\ell} = \int_{C_2} \vec{E}_2\cdot d\vec{\ell} - \int_{C_1} \vec{E}_1\cdot d\vec{\ell}$$

limit where $h \to 0$

where $d\vec{\ell} = \hat{t}\, d\ell$, $\hat{t}$ unit tangent vector along $\mathcal{C}$

$$\therefore \quad \vec{E}_2\cdot\hat{t} = \hat{E}_1\cdot\hat{t}$$

or equivalently: $$\boxed{(\vec{E}_2 - \vec{E}_1)\wedge\hat{n} = 0}$$

1.6 Conductors and insulators

A conductor is an object with a large number of free electrons (charge carriers).

Conductors respond to external electric fiels:
the free charges move around the material as long as they experience a force until they reach equilibrium

Two important consequences

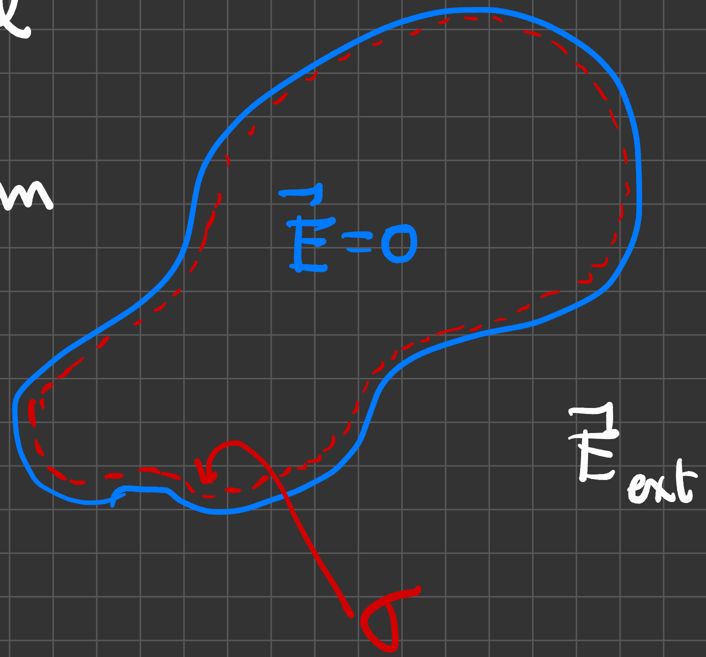
1. In an static situation, inside a conductor

$$\vec{E} = 0$$

because if it weren't then by Coulomb's law the free electrons would continue to move

(2) Suppose a conductor is placed in a region where there is an electric field $\vec{E}_{ext}$.

Then the electrons are forced to move until they reach the surface or the electric field inside reaches its equilibrium and vanishes



One then finds electrons very close to the surface producing a surface charge density σ and they generate an electric field which precisely cancels $\vec{E}_{ext}$

<u>Moreover</u>: as $\vec{E}_{inside} = 0 = -\nabla \Phi_{inside} \Rightarrow \Phi = \text{constant}$ everywhere <u>in</u> the conductor

that is $\boxed{\text{a conductor is an equipotential region of space}}$

An insulator, or dielectric material, on the other hand is an object with charged particles strongly bounded to molecules.

In the presence of an external electric field, charged particles shift positions in response but do not leave the vicinity of their molecules

1.7 Energy stored in the electric field

Let q_1 be a point charge at $\vec{r}_1$.
The electrostatic potential $\Phi_1(\vec{r})$ due to this charge is

$$\Phi_1(\vec{r}) = \frac{q_1}{4\pi\epsilon_0} \frac{1}{|\vec{r}-\vec{r}_1|} \quad \text{at } \vec{r} \neq \vec{r}_1$$

Consider now another point charge q_2 at infinity.
In moving this charge from infinity to a point $\vec{r}_2$
we find that the work done in moving the charge against the electric field $\vec{E}_1$ due to q_1 is

$$W_2 = q_2 \Phi_1(\vec{r}_2) = \frac{q_2}{4\pi\epsilon_0} \frac{q_1}{|\vec{r}_2-\vec{r}_1|}$$

Now move another charge q_3 from infinity to $\vec{r}_3$.

The work done now against the electric field $\vec{E}_1 + \vec{E}_2$ due to the charges q_1 & q_2

$$W_3 = \frac{q_3}{4\pi\varepsilon_0}\left(\frac{q_1}{|\vec{r}_3-\vec{r}_1|} + \frac{q_2}{|\vec{r}_3-\vec{r}_2|}\right)$$

$$= q_3\left(\Phi_1(\vec{r}_3) + \Phi_2(\vec{r}_3)\right)$$

We can continue bringing in charges $q_4, \dots q_N$ from infinity and clearly the <u>total</u> work done is

$$W = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{N} \sum_{j<i} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|} = \frac{1}{2} \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{N} \sum_{j\neq i} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}$$

$$= \frac{1}{2} \sum_{i=1}^{N} q_i \Phi_i$$

For a continuous distribution $\rho(\vec{r})$, the work done is

$$W = \frac{1}{2} \int_V \rho(\vec{r}) \, \Phi(\vec{r}) \, d^3x$$

potential energy per unit charge stored in the electric field $\vec{E} = -\nabla \Phi$

We can write this potential energy in terms of $\vec{E}$.
Use Gauss' law to eliminate ρ:

$$\rho\,\Phi = \epsilon_0\,\Phi\,\nabla\cdot\vec{E} = \epsilon_0\left(\nabla\cdot(\Phi\vec{E}) - \nabla\Phi\cdot\vec{E}\right)$$
$$= \epsilon_0\left(\nabla\cdot(\Phi\vec{E}) + |\vec{E}|^2\right)$$

Then

$$W = \frac{\epsilon_0}{2}\left\{\int_{\partial V}\Phi\vec{E}\cdot d\vec{S} + \int_V |\vec{E}|^2\,dV\right\}$$

Taking V = a very large ball of radius r
and ∂V a sphere

$\Rightarrow$ first term vanishes when $\Phi \sim \frac{c}{r}$ as $r\to\infty$

Hence,

$$W = \frac{\epsilon_0}{2} \int_{\mathbb{R}^3} |\vec{E}|^2 \, dV$$

We can identify the integrand as the energy density

$$\boxed{u = \frac{\epsilon_0}{2} |\vec{E}|^2}$$

Next: Boundary value problem in electrostatics