ELECTROMAGNETISM (PART B)

Chapter 2: Boundary value problems in electrostatics (Part 1)

Lecture 4

2 Boundary value problems in electrostatics

Many problems in electrostatics involve boundary surfaces on which Φ or $\vec{E}$ are given.

$\boxed{2.1}$ Boundary value problems

Want: Solutions of Poisson's eq

$$\nabla^2 \Phi = -\frac{1}{\epsilon_0} \rho$$

in a bounded region of space V involving boundary conditions on $S = \partial V$

What are the appropriate boundary conditions for Poisson's eq (or Laplace's), that is, boundary conditions such that we obtain unique & well defined solutions for a given configuration of charges inside V.

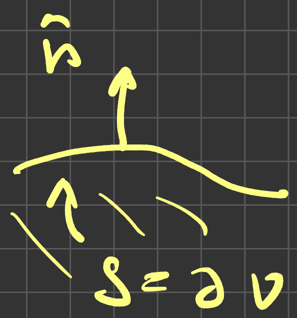
There are two natural boundary conditions

1. Dirichlet boundary conditions

   Φ specified on the boundary surface

2. Newmann boundary conditions

   $\vec{E}\cdot\hat{n}$ specified on the boundary surface.



$\hookrightarrow \vec{E}\cdot\hat{n} = -\nabla\Phi\cdot\hat{n} = -\dfrac{\partial\Phi}{\partial n}$ normal derivative of Φ

One can show uniqueness of the solution of Poisson's eq inside a volume V subject to Dirichlet or Newmann boundary condions on the closing boundary surface $S = \partial V$

Sketch of the proof: See Jackson (Chapter 1)

Suppose there are two solutions Φ_1 & Φ_2 which satisfy the same boundary conditions

Let
$$\psi = \Phi_2 - \Phi_1 .$$

Then
$$\nabla^2 \psi = \underbrace{\nabla^2 \Phi_2}_{-\frac{1}{\epsilon_0}\rho} - \underbrace{\nabla^2 \Phi_1}_{-\frac{1}{\epsilon_0}\rho} = 0 \quad \text{inside } V$$

so ψ satisfies Laplace's eq.

and
$$\begin{cases} \psi = 0 & \text{on } S \quad \text{for Dirichlet b.c} \\ \text{or} & \\ \dfrac{\partial \psi}{\partial n} = 0 & \text{on } S \quad \text{for Neumann b.c} \end{cases}$$

Recall 1st Green's indentity:

u, v arbitrary

$$\int_V (u \nabla^2 v + \nabla u \cdot \nabla v)\, dV = \int_{S=\partial V} u \frac{\partial v}{\partial n}\, dS$$

comes from the div. theorem for $u \nabla v$

$\nabla \cdot (u \nabla v) = u \nabla^2 v + \nabla u \cdot \nabla v$ for LHS

For RHS $u \frac{\partial v}{\partial n} = u \nabla v \cdot \hat{n}$

Apply to $u = v = \psi$:

$$\int_V (\underbrace{\psi \nabla^2 \psi}_{\to 0} + \nabla \psi \cdot \nabla \psi)\, dV = \int_{S=\partial V} \underbrace{\psi \frac{\partial \psi}{\partial n}}_{\to 0}\, dS$$

as $\psi|_S = 0$ for Dirichlet

$\frac{\partial \psi}{\partial n}\Big|_S = 0$ for Neumann

Then $\int_V |\nabla \Psi|^2 dV = 0$ for both types of boundary condition

$\Rightarrow$ <u>inside</u> V : $\nabla \Psi = 0$ ie $\Psi =$ constant

$= \Phi_2 - \Phi_1$

Dirichlet : $\Psi|_S = 0 \Longrightarrow \Phi_1 = \Phi_2$ inside V

Neumann : $\frac{\partial \Psi}{\partial n}\Big|_S = 0 \Longrightarrow \Phi_1 = \Phi_2 +$ constant inside V

Up to an unphysical constant, the solution inside V, with $S = \partial V$ a closed surface is unique.

Remark: One still has a unique solution for a problem with "mixed" boundary conditions ie Dirichlet over parts of V and Neumann over other parts of V.

BUT for a closed surface there does not exist a solution if both types of boundary conditions are imposed on a closed surface

2.2 Green's functions:

We can write down a solution of Poisson's eq using Green's functions

Definition: a Green's function is a function $G(\vec{r}, \vec{r}\,')$ which satisfies

$$\boxed{\nabla'^2 G(\vec{r}, \vec{r}\,') = -4\pi\,\delta(\vec{r} - \vec{r}\,')}$$

$\nabla'^2 G(\vec{r}, \vec{r}\,') = 0$
$\forall\ \vec{r} \neq \vec{r}\,'$

The solutions of this equation are not unique, but we already know of a solution!

notice the symmetry $\vec{r} \leftrightarrow \vec{r}\,'$

$$G(\vec{r}, \vec{r}\,') = \frac{1}{|\vec{r} - \vec{r}\,'|}$$

which is $4\pi\epsilon_0$ (potential at $\vec{r}$ due to a unit charge at $\vec{r}\,'$)

This means that any solution is of the form

$$G(\vec{r}, \vec{r}\,') = \frac{1}{|\vec{r}-\vec{r}\,'|} + F(\vec{r}, \vec{r}\,')$$

where $$\nabla'^2 F(\vec{r}, \vec{r}\,') = 0 \text{ in } V$$

Interpret F as the potential due to a distribution of charges which lie $\boxed{\text{outside}}$ V.

Next use this to write a solution of

$$\nabla^2 \Phi = -\frac{1}{\epsilon_0}\rho$$

Recall Green's theorem (Green's 2nd identity)

$$\int_V \left[u(\vec{r}\,')\, \nabla'^2 v(\vec{r}\,') - v(\vec{r}\,')\, \nabla'^2 u(\vec{r}\,') \right] dV'$$

u, v arbitrary

$$= \int_{S=\partial V} \left[u(\vec{r}\,') \frac{\partial v(\vec{r}\,')}{\partial n'} - v(\vec{r}\,') \frac{\partial u(\vec{r}\,')}{\partial n'} \right] dS'$$

[from Green's 1st id : Greens 1st u & v − Green's 1st v & u]

Consider Green's theorem with
$u = \Phi(r)$ electrostatic potential
$v = G(\vec{r}, \vec{r}\,')$ Green's function

Then

$$\int_V \Big[\Phi(\vec r\,') \underbrace{\nabla'^2 G(\vec r, \vec r\,')}_{-4\pi\delta(\vec r - \vec r\,')} - G(\vec r, \vec r\,') \underbrace{\nabla'^2 \Phi(\vec r\,')}_{-\frac{1}{\epsilon_0}\rho(\vec r\,')} \Big] dV'$$

Φ sln of Poisson's eq

$$= \iint_{S=\partial V} \Big[\Phi(\vec r\,') \frac{\partial G(\vec r, \vec r\,')}{\partial n'} - G(\vec r, \vec r\,') \frac{\partial \Phi(\vec r\,')}{\partial n'} \Big] dS'$$

$$\text{LHS} = -4\pi \Phi(\vec r) + \frac{1}{\epsilon_0} \int_V G(\vec r, \vec r\,') \rho(\vec r\,') \, dV'$$

Then

$$\Phi(\vec r) = \frac{1}{4\pi\epsilon_0} \int_V G(\vec r, \vec r\,') \rho(\vec r\,') dV' + \frac{1}{4\pi} \int_{S=\partial V} \Big[G(\vec r, \vec r\,') \frac{\partial \Phi(\vec r\,')}{\partial n'} - \Phi(\vec r\,') \frac{\partial G(\vec r, \vec r\,')}{\partial n'} \Big] dS'$$

Then

given

$$\begin{cases} \Phi(\vec{r}) = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int_V G(\vec{r},\vec{r}')\,\rho(\vec{r}')\,dV' \\ \qquad + \dfrac{1}{4\pi}\displaystyle\int_{S=\partial V}\left[G(\vec{r},\vec{r}')\dfrac{\partial\Phi(\vec{r}')}{\partial n'} - \Phi(\vec{r}')\dfrac{\partial G(\vec{r},\vec{r}')}{\partial n'}\right]dS' \end{cases}$$

given on S for Neumann b.c

given on S for Dirichlet bound. cond.

Remarks: $G(\vec{r},\vec{r}') = \dfrac{1}{|\vec{r}-\vec{r}'|} + F(\vec{r},\vec{r}')$

need this!

strategy: use the freedom to choose F to eliminate one or the other surface integrals

For Dirichlet boundary conditions, where Φ is specified on $S = \partial V$, we demand

$$G_D(\vec{r}, \vec{r}') = 0 \qquad \forall\ \underline{\vec{r}}' \text{ on } S$$

Then

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V G(\vec{r}, \vec{r}')\, \rho(\vec{r}')\, dV' - \frac{1}{4\pi}\int_{S=\partial V} \Phi(\vec{r}')\, \frac{\partial G(\vec{r}, \vec{r}')}{\partial n'}\, dS'$$

given

given on S

general soln but need to find

$$\nabla'^2 G_D(\vec{r}, \vec{r}') = 0$$

st $\quad G_D(\vec{r}, \vec{r}')\Big|_S = 0$

For Newmann boundary conditions,
where $\frac{\partial \Phi}{\partial n}$ is specified on S,
one can try a similar trick

$$\frac{\partial}{\partial n'} G_N(\vec{r}, \vec{r}') = 0 \quad \text{on} \quad S$$

This is inconsistent. Recall $\nabla'^2 G(\vec{r}, \vec{r}') = -4\pi \delta(\vec{r} - \vec{r}')$
Integrating the LHS

$$\int_V \nabla'^2 G(\vec{r}, \vec{r}')\, dV' = \int_V \nabla' \cdot \nabla' G(\vec{r}, \vec{r}')\, dS'$$

$$= \int_{S = \partial V} \nabla' G(\vec{r}, \vec{r}') \cdot d\vec{S}' = \int_S \frac{\partial G(\vec{r}, \vec{r}')}{\partial n'}\, dS'$$

Integrating the RHS:

$$-4\pi \int_V \delta(\vec{r} - \vec{r}')\, dV' = -4\pi, \quad \vec{r} \in V$$

But we have just shown that

$S = \partial V$

$$\int_S \frac{\partial G_N(\vec{r}, \vec{r}\,')}{\partial n'} \, dS' = -4\pi$$

Require instead,

$$\left. \frac{\partial G_N(\vec{r}, \vec{r}\,')}{\partial n'} \right|_S = -\frac{4\pi}{A}$$

A total area of S

Recalling:

general formal soln

given

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V G_N(\vec{r},\vec{r}')\,\rho(\vec{r}')\,dV' + \frac{1}{4\pi}\int_{S=\partial V}\left[G_N(\vec{r},\vec{r}')\underbrace{\frac{\partial\Phi(\vec{r}')}{\partial n'}}_{\text{given } S} - \Phi(\vec{r}')\underbrace{\frac{\partial G_N(\vec{r},\vec{r}')}{\partial n'}}_{= -4\pi/A}\right]ds'$$

The formal solution for Neumann boundary conditions on $S=\partial V$ is

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V G_N(\vec{r},\vec{r}')\,\rho(\vec{r}')\,dV' + \frac{1}{4\pi}\int_{S=\partial V} G_N(\vec{r},\vec{r}')\frac{\partial\Phi(\vec{r}')}{\partial n'}\,ds' + \langle\Phi_N\rangle$$

with $\langle\Phi_N\rangle \equiv \dfrac{1}{A}\displaystyle\int_{S=\partial V}\Phi(\vec{r}')\,ds'$ average of Φ per unit area over the whole surface S

and $G_N(\vec{r},\vec{r}')$ st $\nabla'^2 G_N(\vec{r},\vec{r}') = -4\pi\,\delta(\vec{r}-\vec{r}')$

with b.c.s $\left.\dfrac{\partial G_N(\vec{r},\vec{r}')}{\partial n'}\right|_S = -\dfrac{4\pi}{A}$

Remarks:

* In practice it is hard to determine $G(\vec{r}, \vec{r}')$ because it depends on the shape of S

* One can choose $G(\vec{r}, \vec{r}') = G(\vec{r}', \vec{r})$

  (For $G(\vec{r}, \vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|}$ true !)

  - Dirichlet boundary conditions:

    This is not a choice, it can be proven

    (use Green's theorem with $u = G_D(\vec{r}, \vec{r}')$ $v = G(\vec{r}', \vec{r})$)

  - Neumann boundary conditions:

    this is not automatic, but can be imposed separately.

2 Boundary value problems in electrostatics

2.1 Boundary value problems

2.2 Green's functions.

Next

- Method of images → & its relation to Green's functions
- examples
- orthonormal functions to find Green's functions

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