ELECTROMAGNETISM (PART B)

Chapter 2: Boundary value problems in electrostatics (Part 2)

Lecture 5

$\boxed{2}$ Boundary value problems in electrostatics

$\boxed{2.1}$ Boundary value problems

$\boxed{2.2}$ Green's functions.

Today

$\boxed{2.3}$ Method of images and relation to Green's functions

(examples

next lecture

⋮

on functions to find Green's functions

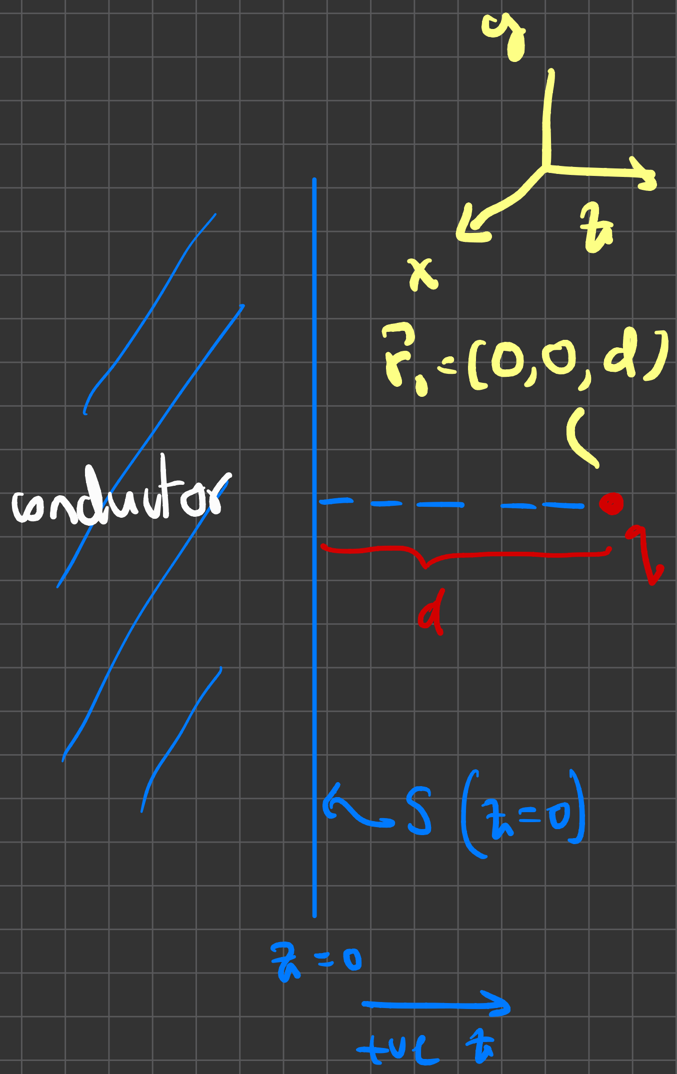
2.3 Method of images

Useful when there is a discrete distribution of point charges in the presence of boundary surfaces

Idea: given a distribution of charges in a region V & boundary conditions corresponding to the presence of (perhaps charged) surface boundaries, place an appropriate distribution of charges outside V ($\mathbb{R}^3 \setminus V$) (image charges) which simulate the boundary conditions.

$\Rightarrow$ solve a problem on $\mathbb{R}^3$ without boundaries and then constrain it back to V.

Example: Consider an infinite conducting plane & place a charge q at $\vec{r}_0$.



Want: Φ (and $\vec{E}$) in region $z>0$

ie solution of $\nabla^2 \Phi = -\frac{1}{\epsilon_0}\rho$, $\forall\, \vec{r} \in V$

where $V = \{\vec{r} \in \mathbb{R}^3 : z > 0\}$

st $\Phi\big|_S = 0$ (Dirichlet boundary condition)

why?

$$\rho = q\,\delta(\vec{r} - \vec{r}_0)$$
$$= q\,\delta(x)\,\delta(y)\,\delta(z-d)$$

because S a conductor $\Rightarrow$ S is an equipotential $\Rightarrow \Phi\big|_S = \text{const}$ so choose $\Phi = 0$

method of images: infer from the geometry an appropriate distribution of image point charges outside V ($z<0$ in our example) st they produce a potential with the required boundary conditions.

problem in a region V with boundary conditions on $S=\partial V$ ie

$$\nabla^2 \Phi = -\frac{1}{\epsilon_0}\rho \quad \forall \vec{r} \in V$$

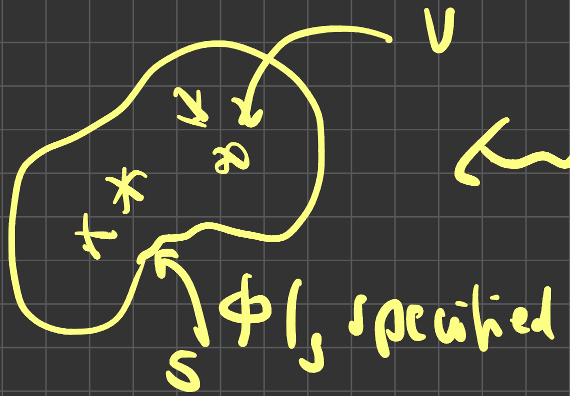
st $\Phi|_S$ is specified

$\longleftrightarrow$

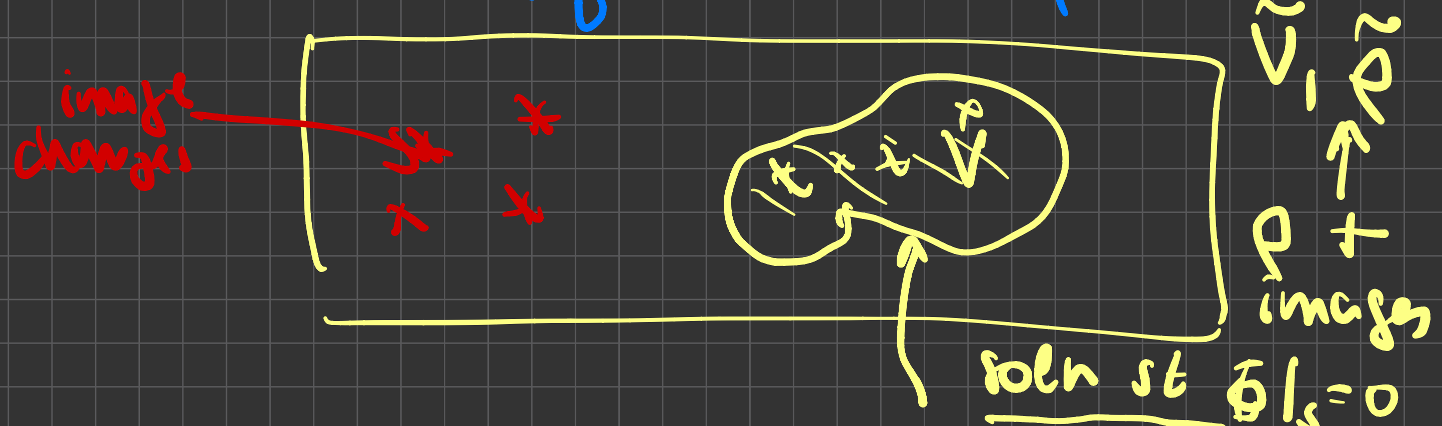
problem in an enlarged region $\tilde{V}$, $\mathbb{R}^3$, $V \subset \tilde{V}$ + image charges external to V (in $\tilde{V} \setminus V$) st

$$\nabla^2 \Phi = -\frac{1}{\epsilon_0}\tilde{\rho} \quad \forall \tilde{r} \in \tilde{V}$$

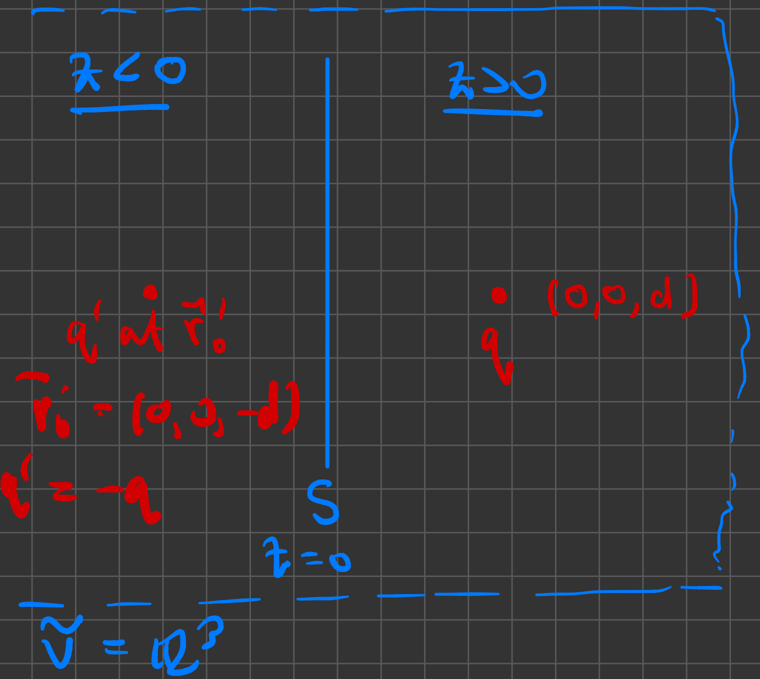
with no boundary condition on S $\boxed{\text{but}}$ st Φ takes the values of the boundary conditions specified on S



$\longleftrightarrow$



returning to our example: $\tilde{V} = \mathbb{R}^3$



place a charge q' in region $z<0$ at $\vec{r}_0'$ and solve

$$\nabla^2 \Phi = -\frac{1}{\epsilon_0}\left(q\,\delta(\vec{r}-\vec{r}_0) + q'\,\delta(\vec{r}-\vec{r}_0')\right)$$

st solution Φ in region $z>0$ satisfies $\Phi\big|_{z=0} = 0$

It is an easy guess

$$q' = -q \;, \qquad \vec{r}_0' = (0,0,-d)$$

Then, the total potential in the enlarged region ($\mathbb{R}^3$) is

$$\Phi(\vec{r}) = \frac{q}{4\pi\epsilon_0}\frac{1}{|\vec{r}-\vec{r}_0|} - \frac{q}{4\pi\epsilon_0}\frac{1}{|\vec{r}+\vec{r}_0|}$$

Clearly:

(1) $$\Phi \Big|_{z=0} = 0$$ (easy)

(2) $$\nabla^2 \Phi = -\frac{q}{\epsilon_0}\delta(x)\delta(y)\delta(z-d) + \underbrace{\frac{q}{\epsilon_0}\delta(x)\delta(y)\delta(z+d)}_{\to 0 \text{ for } z>0}$$

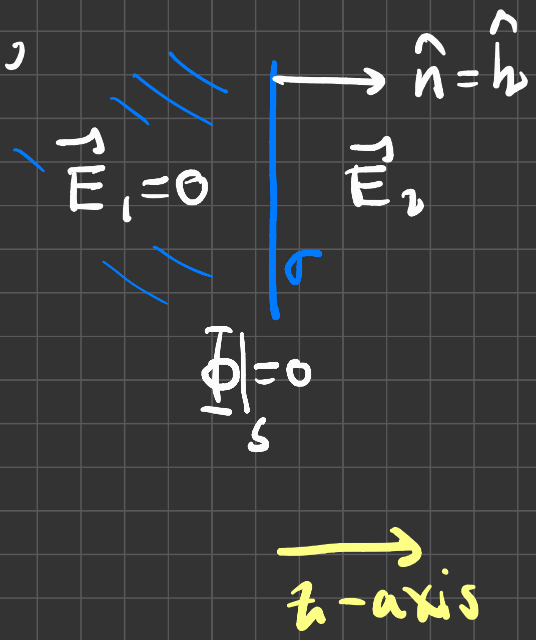
So Φ satisfies Poisson's eq in V $(z>0)$

Question: What is the charge density σ on the surface S induced by the point charge q at $\vec{r}_0=(0,0,d)$?

There is a discontinuity of $\frac{1}{\epsilon_0}\sigma$ in the normal component to S of $\vec{E}$ in crossing S,

$$(\vec{E}_2-\vec{E}_1)\cdot\hat{n}\Big|_S=\frac{1}{\epsilon_0}\sigma \;\Rightarrow\; \underbrace{\vec{E}_2\cdot\hat{z}\Big|_S}_{-\frac{\partial \Phi}{\partial z}\Big|_S}=\frac{1}{\epsilon_0}\sigma$$

$S \leftarrow z=0$



$\vdots$

$$\sigma=-\frac{qd}{2\pi}\frac{1}{[x^2+y^2+d^2]^{3/2}}$$

- symmetric around $x=y=0$
- max at $x=y=0$
- $\sigma\to 0$ at large dist. from origin

What is the $\underline{\underline{\text{total}}}$ charge on the surface of the conductor?

$$\int_S \sigma \, dS = -\frac{qd}{2\pi} \iint_{-\infty}^{\infty} \frac{1}{(x^2+y^2+d^2)^{3/2}} \, dx \, dy = \cdots = -q$$

charge of the image!

Force on the charge q

$$F_z = -q \left. \frac{\partial \Phi}{\partial z} \right|_{z=d} = \cdots = \frac{1}{4\pi\epsilon_0} \frac{q^2}{(2d)^2},$$

$\vec{F}$ attractive in the z-direction

$\rightarrow$ Coulomb force on q due to the charge $q' = -q$ at $\vec{r}_0{}' = (0,0,-d)$.

Can you generalize this result to the case where there is a distribution of N point charges in the region $z > 0$?

Relation between the method of images and Green's functions

Try to solve the same example using Green's functions

For a boundary value problem with Dirichlet boundary conditions recall

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{r}\,')\, G_D(\vec{r},\vec{r}\,')\, dV' - \frac{1}{4\pi}\int_S \underbrace{\Phi(\vec{r}\,')}_{0} \frac{\partial G_D}{\partial n'}(\vec{r},\vec{r}\,')\, dS'$$

$\vec{r} \in V$

$q\,\delta(x')\delta(y')\delta(z'-d)$

as $\Phi\big|_S = 0$

$\nabla'^2 G_D(\vec{r},\vec{r}\,') = -4\pi\delta(\vec{r}-\vec{r}\,')$
st $G_D(\vec{r},\vec{r}\,') = 0 \;\forall\, \vec{r}\,' \in S$
(our example S = surface $z=0$)

$$\Rightarrow \Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} q\, G_D(\vec{r},\vec{r}_0)\,, \quad \vec{r}_0 = (0,0,d)\;;\; \forall\, \vec{r} \in V$$

For the Dirichlet Green's function:

$$G_D(\vec{r},\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|} + F_D(\vec{r},\vec{r}'), \quad \nabla'^2 F_D(\vec{r},\vec{r}') = 0$$

inside V

$F_D \sim$ potential of a system of charges outside V

with $\left. G_D(\vec{r},\vec{r}') \right|_{z'=0} = 0$

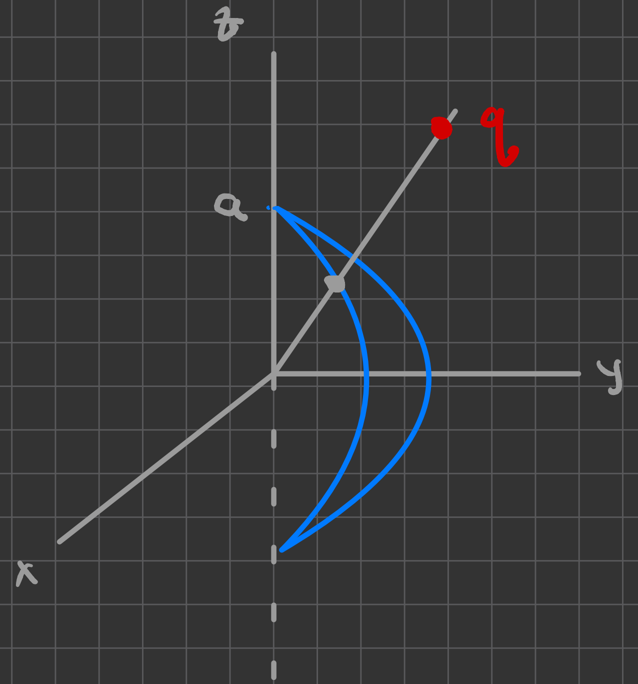
But this is the problem we just solved using the method of images!

Clearly, one can choose $F_D(\vec{r},\vec{r}') = -\frac{1}{|\vec{r}-\vec{r}_0'|}$, $\vec{r}_0' = (x', y', -z')$

Indeed: $\nabla'^2 F_D(\vec{r},\vec{r}') = 0$ when $z'>0$

Finally: $\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{1}{|\vec{r}-\vec{r}_0|} - \frac{1}{|\vec{r}-\vec{r}_0'|}\right)$ $\quad \vec{r}_0' = (0,0,-d)$

Example: Consider a charge q placed at a point in space outside a conducting grounded sphere of radius a



Charge q located at a distance $b > a$ from the center of sphere

Find the electrostatic potential $\Phi(\vec{r})$, $\forall \vec{r}$ with $|\vec{r}| > a$ st

$$\Phi(\vec{r})\Big|_{r=a} = 0$$

Try first the method of images

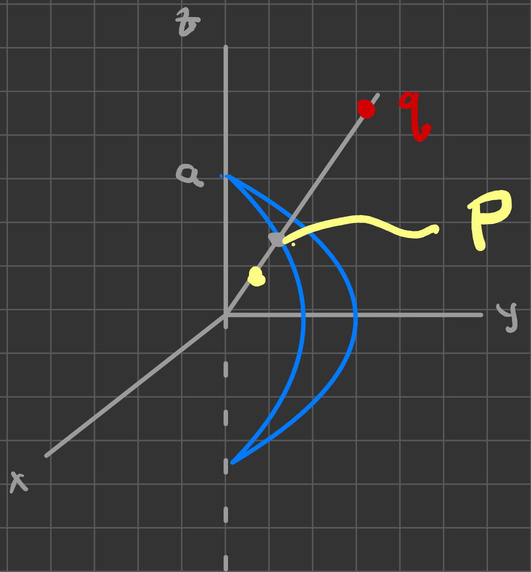
Assume that only one image, charge q', is needed inside the sphere

By symmetry: the image charge lies on the line joining the origin to the point where the charge q is

The Coulomb potential due to the charges is

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left\{ q \frac{1}{|\vec{r} - b\,\hat{n}_P|} + q' \frac{1}{|\vec{r} - b'\,\hat{n}_P|} \right\}$$

where



- $b\,\hat{n}_P$ is the position of q
  - $\hat{n}_P$: unit vector normal to S at P
- $b'\hat{n}_P$ is the position of q'

We must find q' & b' st $\Phi(\vec{r})\Big|_{r=a} = 0$

let $\vec{r} = r\hat{n}$, $\hat{n}$ = unit vector

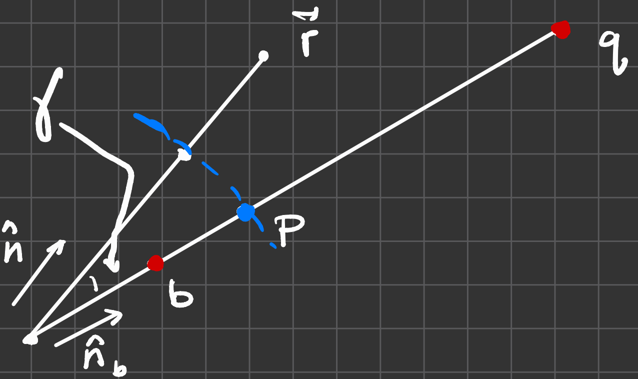
On S :

$$\Phi(a\hat{n}) = 0 \iff 0 = \frac{q}{|a\hat{n} - b\hat{n}_p|} + \frac{q'}{|a\hat{n} - b'\hat{n}_p|} \text{ for } \boxed{\text{any } \hat{n}} \text{ on } S$$

$$\iff q|a\hat{n} - b'\hat{n}_p| = -q'|a\hat{n} - b\hat{n}_p| \quad \forall\, \hat{n} \text{ on } S$$

$$\iff q\sqrt{1 - \frac{2b'}{a}\cos\gamma + \frac{b'^2}{a^2}} = -q\sqrt{1 - \frac{2b}{a}\cos\gamma + \frac{b^2}{a}}, \quad \forall\, \gamma$$

$$\hat{n}\cdot\hat{n}_p = \cos\gamma$$



$$\Phi(a\hat{n}) = 0 \iff q' = -q\frac{a}{b}, \quad b' = \frac{a^2}{b} < a$$

$a \cdot \frac{a}{b}$

For the last step:

$$q^2\left(1 - \frac{2b'}{a}\cos\gamma + \frac{b'^2}{a^2}\right) = q'^2\left(1 - \frac{2b}{a}\cos\gamma + \frac{b^2}{a^2}\right)$$

$$q^2\left(1 + \frac{b'^2}{a^2}\right) - q'^2\left(1 + \frac{b^2}{a}\right) = \frac{2}{a}\cos\gamma\left(q^2 b' - q'^2 b\right)$$ match coeffs of $\cos\gamma$

(2) $q^2\left(1 + \frac{b'^2}{a^2}\right) = q'^2\left(1 + \frac{b^2}{a^2}\right)$ (1) $q'^2 = \frac{b'}{b} q^2$

(1) $\longrightarrow$ (2): $1 + \frac{b'^2}{a^2} = \frac{b'}{b}\left(1 + \frac{b^2}{a^2}\right)$ is a quadratic eq for b'

... $\left(\frac{b'}{b} - 1\right)\left(\frac{b'}{b} - \left(\frac{a}{b}\right)^2\right) = 0 \Rightarrow b' = b$ or $b' = \frac{a^2}{b}$

no! this is outside S

back into (1) $q'^2 = \frac{a^2}{b^2} q^2 \Rightarrow q' = -\frac{a}{b} q$

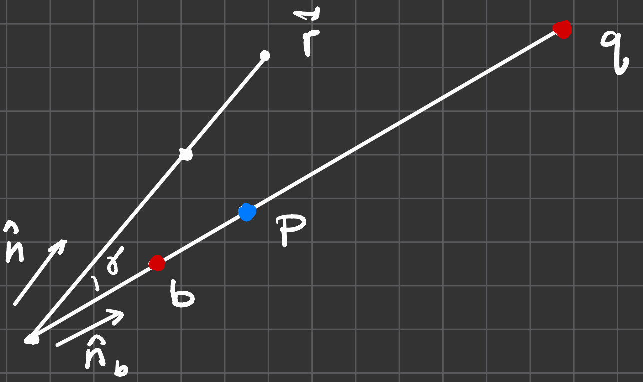
Then:

$$\Phi(\vec{r}) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\vec{r}-b\hat{n}_p|} - \frac{a}{b}\frac{1}{|\vec{r}-\frac{a^2}{b}\hat{n}_p|}\right)$$

$$= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{(r^2+b^2-2br\cos\gamma)^{1/2}} - \frac{a}{b}\frac{1}{\left(r^2+\left(\frac{a^2}{b}\right)^2-2\frac{a^2}{b}r\cos\gamma\right)^{1/2}}\right)$$

Charge induced on the surface of the sphere

$$\sigma = -\epsilon_0 \frac{\partial\Phi}{\partial n}\bigg|_{r=a} = \dots = -\frac{q}{4\pi}\frac{b^2}{a}\frac{1-a^2/b^2}{(a^2+b^2-2ab\cos\gamma)^{3/2}}$$



Total induced surface charge

$$Q = \int_S \sigma\, ds = \dots = -q\frac{a}{b}$$

as expected

General solution to the Dirichlet problem on the sphere

let V st $\partial V = S$ a sphere of radius a

with $\Phi(\vec{r})\Big|_{r=a} = V(\theta, \phi)$ a given function.

Recall:

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{r}\,')\, G_D(\vec{r}, \vec{r}\,')\, dV' - \frac{1}{4\pi}\int_{\partial V = S} \underbrace{V(\theta', \phi')}_{\Phi|_{r'=a}} \frac{\partial G_D(\vec{r}, \vec{r}\,')}{\partial n'}\, dS'$$

where $\nabla'^2 G_D(\vec{r}, \vec{r}\,') = -4\pi\,\delta(\vec{r} - \vec{r}\,')$; $G_D(\vec{r}, \vec{r}\,') = 0$, $r' = a$

Then $G_D(\vec{r}, \vec{r}\,') = \frac{1}{|\vec{r} - \vec{r}\,'|} \boxed{- \frac{a}{r'} \frac{1}{\left|\vec{r} - \frac{a^2}{r'^2}\vec{r}\,'\right|}}$

F_D = potential for a distribution of charges inside S st $G_D(\vec{r}, \vec{r}\,')\big|_{r'=a} = 0$

$\Longrightarrow$ F_D potential for a charge $q' = -\frac{a}{r'}$ at $\frac{a^2}{r'^2}\vec{r}\,'$

In spherical coordinates

$$G_D(\vec{r},\vec{r}') = \frac{1}{(r^2+r'^2-2rr'\cos\gamma)^{1/2}} - \frac{1}{\left(\frac{1}{a^2}r'^2r^2+a^2-2rr'\cos\gamma\right)^{1/2}}$$

where $\vec{r}\cdot\vec{r}' = rr'\cos\gamma$

Note the symmetry $G_D(\vec{r},\vec{r}') = G_D(\vec{r}',\vec{r})$

We can finally write Φ:

Compute $$\left.\frac{\partial G_D}{\partial n'}\right|_{r'=a} = \frac{r^2-a^2}{a(r^2+a^2-2ar\cos\gamma)^{3/2}}$$

$$\boxed{\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{r}')G_D(\vec{r},\vec{r}')\,dV' + \frac{1}{4\pi}\int_S V(\theta',\phi')\frac{(r^2-a^2)}{a(r^2+a^2-2ar\cos\gamma)^{3/2}}\,dS'}$$

12 Boundary value problems in electrostatics

12.1 Boundary value problems ✓

12.2 Green's functions ✓

This lecture

12.3 Method of images & its relation to Green's functions ✓

examples.

Next lecture

12.4 on functions to find Green's function