ELECTROMAGNETISM (PART B)

Chapter 2: Boundary value problems in electrostatics (Part 3)

Lecture 6

$\boxed{2}$ Boundary value problems in electrostatics

$\boxed{2.1}$ Boundary value problems ✓

$\boxed{2.2}$ Green's functions ✓

$\boxed{2.3}$ Method of images & its relation to Green's functions ✓

examples.

This lecture

$\boxed{2.4}$ Method of orthonormal functions

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$\boxed{2.4.1}$ Generalities

- powerful technique
- representations of solutions of potential problems by expansions in orthogonal functions

Idea: an arbitrary function $f(x)$ which is square integrable on (a,b) $\left(\int_a^b |f(x)|^2 dx < \infty\right)$ can be expanded in a series of a complete set of orthonormal functions

what is this? next.

orthonormality :

Consider an interval (a,b) and a set of (real or complex) functions $u(x)$ of one variable

$$\{ u_n(x) , \quad n = 1, 2, \ldots , \quad x \in (a,b] \}$$

which are orthonormal

$$\int_a^b u_n^*(x)\, u_m(x)\, dx = 1\, \delta_{nm}$$

← normalized to 1

We represent a function $f(x)$ by a series

$$\bar{f}(x) = \sum_{n=1}^{\infty} a_n u_n(x)$$

where the coefficients a_n are given by

by the ON property $\{u_n\}$ $\rightarrow$ $$a_n = \int_a^b u_n^*(x)\, f(x)\, dx$$

In fact:

$$\int_a^b u_n^*(x)\, \bar{f}(x)\, dx = \sum_{m=1}^{\infty} a_m \int_a^b u_n^*(x)\, u_m(x)\, dx$$

$$\overset{\text{ON}}{=} \sum_{m=1}^{\infty} a_m\, \delta_{mn} = a_n$$

completeness:

We say that the orthonormal set $\{u_n(x), n=1,\dots\}$ is complete if $\bar{f}$ (as above) converges to f.

Consider

$$a_n = \int_a^b u_n^*(x')f(x')dx'$$

$$\bar{f}(x) = \sum_{n=1} a_n u_n(x) = \sum_{n=1} \int_a^b u_n^*(x') f(x') u_n(x)\, dx'$$

$$= \int_a^b f(x') \left(\sum_{n=1} u_n^*(x') u_n(x) \right) dx'$$

This must be true for $\boxed{\text{any}}$ function on (a,b). Then

$$\sum_{n=1}^{\infty} u_n^*(x') u_n(x) = \delta(x'-x)$$

completeness or closure relation

Remark 1: it is useful to compare orthogonality vs closure

$$\int_a^b u_n^*(x)\, u_m(x)\, dx = \delta_{nm} \quad \text{vs} \quad \sum_{n=1}^{\infty} u_n^*(x')\, u_n(x) = \delta(x'-x)$$

$x \in (a,b)$ a continuous variable $\qquad$ n a discrete variable

Remark 2: one can extend this to higher dimensions. Obvious generalizations, for example to two dimensions (x,y) where

$$x \in (a,b] \qquad y \in (c,d)$$

are orthonormal functions in each dimension $u_n(x)$, $V_m(y)$. Then the expansion of an arbitrary function $f(x,y)$ is $\sum a_{mn}\, u_m(x)\, V_n(y)$

with $a_{mn} = \int_a^b dx \int_c^d dy\; u_m^*(x)\, V_n(y)\, f(x,y)$

one of the most famous examples is the

Example: Fourier series of a periodic function

We can represent a periodic function $f(x)$ with period a.
WLOG: let $x \in \left(-\frac{a}{2}, \frac{a}{2}\right)$.

A complete set of orthonormal functions is
$$\frac{1}{\sqrt{a}}, \ \sqrt{\frac{2}{a}} \sin\left(\frac{2\pi m}{a} x\right), \ \sqrt{\frac{2}{a}} \cos\left(\frac{2\pi m}{a} x\right), \ m \in \mathbb{Z}_+$$

We then represent $f(x)$ by
$$f(x) = \frac{1}{2} A_0 + \sum_{m=1}^{\infty} A_m \cos\left(\frac{2\pi m}{a} x\right) + B_m \sin\left(\frac{2\pi m}{a} x\right)$$

where
$$A_m = \frac{2}{a} \int_{-a/2}^{a/2} f(x) \cos\left(\frac{2\pi m}{a} x\right) dx$$
$$B_m = \frac{2}{a} \int_{-a/2}^{a/2} f(x) \sin\left(\frac{2\pi m}{a} x\right) dx$$

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Example: Fourier series (again)

Start with a complete orthonormal set of complex exponentials

$$\psi_m(x) = \frac{1}{\sqrt{a}} e^{i\left(\frac{2\pi m}{a}\right)x} , \quad m \in \mathbb{Z}$$

on the interval $\left(-\frac{a}{2}, \frac{a}{2}\right)$. Any arbitrary periodic function has the series expansion

$$f(x) = \frac{1}{\sqrt{a}} \sum_{m=-\infty}^{\infty} \alpha_m \, e^{i\left(\frac{2\pi m}{a}\right)x}$$

where $$\alpha_m = \frac{1}{\sqrt{a}} \int_{-a/2}^{a/2} e^{-i\left(\frac{2\pi m}{a}\right)x'} f(x') \, dx'$$

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Fourier integrals : what happens if the function is not periodic?

We can think of the interval as becoming "infinite" & then the set of orthonormal functions $u_n(x)$ may become a continuum of functions

$$\{u_n(x), n = 1, \dots\} \rightsquigarrow u_k(x) \quad k \in \mathbb{R}$$

Kronecker δ function $\rightsquigarrow$ Dirac δ-function in the orthogonality relation let $u_m(x) \rightsquigarrow u_k(x) = \frac{1}{2\pi} e^{ikx}$ $\quad k \leftarrow \frac{2\pi m}{a}$ $\quad x \in \mathbb{R}$

This is a set of complete orthonormal functions

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(k-k')x}\,dx = \delta(k-k'), \quad \underline{\text{orthonormality}}$$

Dirac δ-function

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(x-x')k}\,dk = \delta(x-x'), \quad \underline{\text{completeness}}$$

These are very useful representations of the Dirac δ-function

The Fourier $\boxed{\text{integral}}$ of a function $f(x)$ is

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k)\, e^{ikx}\, dk$$

$\boxed{\text{exponential Fourier integral}}$

where

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx$$

12.4.2 Electrostatic problems with rectangular symmetry

Consider Laplace's equation in Cartesian coordinates

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

look for separable solutions

$$\Phi(x, y, z) = X(x)\, Y(y)\, Z(z)$$

Then

$$YZ \frac{d^2X}{dx^2} + XZ \frac{d^2Y}{dy^2} + XY \frac{d^2Z}{dz^2} = 0 \qquad \times \frac{1}{XYZ}$$

$$\Rightarrow \quad \underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{\text{only on } x} + \underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{\text{only } y} + \underbrace{\frac{1}{Z}\frac{d^2Z}{dz^2}}_{\text{only } z} = 0$$

Each term must be separately a constant!

Then

$$\frac{1}{X(x)}\frac{d^2X}{dx^2} = A \quad , \quad \frac{1}{Y(y)}\frac{d^2Y}{dy^2} = B, \quad \frac{1}{Z(z)}\frac{d^2Z}{dz^2} = C$$

with $\quad A + B + C = 0$

Take for example: $\quad \dfrac{d^2X}{dx^2} - AX = 0$

Then

$$X(x) = e^{\pm i\alpha x} \qquad A = \alpha^2$$

or

$$X(x) = e^{\pm \alpha x} \qquad A = -\alpha^2$$

General solution is a linear combination

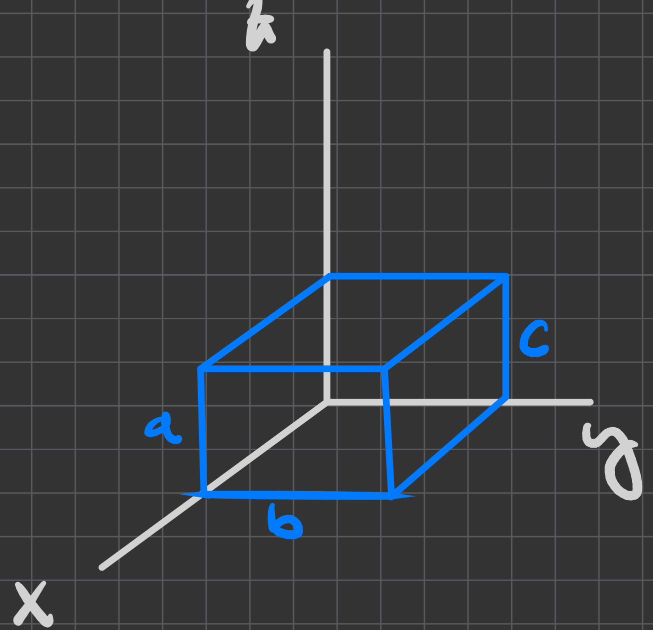
$$\Phi(x, y, t) = e^{\pm i\alpha x} e^{\pm i\beta y} e^{\pm\sqrt{\alpha^2+\beta^2}\, t}$$

when eg $A = \alpha^2$ $B = \beta^2$ $C = -\alpha^2 - \beta^2$

etc ..

Imposing boundary conditions on the general solution $\Rightarrow$ restrictions on the values A, B & the coefficients in the series.

Example: Consider a rectangular box with dimensions (a, b, c)



Suppose $\Phi = 0$ on all surfaces except $z = c$

where $\Phi(x, y, c) = V(x, y)$

The general solution is a linear combination of

$\boxed{\begin{array}{l} A = \alpha^2 \\ B = \beta^2 \end{array}}$

$$e^{\pm i\alpha x} \, e^{\pm i\beta y} \, e^{\pm\sqrt{\alpha^2+\beta^2}\, z}$$

[If instead $X(x) = (c_1 \cosh \alpha x + c_2 \sinh \beta x)$

$\Phi(x=0) = 0 \implies c_1 = 0$

$\Phi(x=a) = 0 \implies c_2 \sinh \beta a = 0 \Rightarrow c_2 = 0$)]

consider first the boundary conditions at $x=0$, $y=0$, and $z=0$ where

$$\Phi\big|_{x=0}=0 \qquad \Phi\big|_{y=0}=0 \qquad \Phi\big|_{z=0}$$

$$\Rightarrow \quad \sin\alpha x \, \sin\beta y \, \sinh\left(\sqrt{\alpha^2+\beta^2}\, z\right)$$

no cosines ← no cosh

Next consider: $x=a$ & $y=b$ where

$$\Phi\big|_{x=a}=0 \quad \& \quad \Phi\big|_{y=b}=0$$

- $x=a$: terms $\sin\alpha a \sin\beta y \sinh(\sqrt{\alpha^2+\beta^2}\, z)$

$$\Rightarrow \quad \alpha a = m\pi, \; m\in\mathbb{Z} \qquad \text{as } \sin\alpha a = 0$$

- similarly for $y=b$: $\beta b = n\pi$, $n\in\mathbb{Z}$

So far:

$$\boxed{\Phi(x,y,z) = \sum_{m,n=1} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \sinh\left(\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}\, z\right)}$$

Finally: require $\Phi\Big|_{z=c} = V(x,y)$

$$\Rightarrow V(x,y) = \sum_{m,n=1} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \sinh\left(\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}\, c\right)$$

$(0,a)$ $(0,b)$

↳ sine Fourier series expansion for $V(x,y)$

Hence: the coefficients A_{mn} are given by (exercise)

$$\boxed{A_{m,n} = \frac{4}{ab\sinh\left(\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{c^2}}\, c\right)} \int_0^a\int_0^b \sin\left(\frac{m\pi a}{x}\right)\sin\left(\frac{n\pi y}{b}\right) V(x,y)\, dx\, dy}$$

- caution: one needs to use an odd extension of $V(x,y)$ to $(-a,a)$ & $(-b,b)$

Compare with solution for Φ in terms of a Green's function. For Dirichlet boundary conditions

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \cancelto{0}{\rho(\vec{r}')}\, G_D(\vec{r},\vec{r}')\,dV' - \frac{1}{4\pi}\int_{\underset{z=0}{S}} \underbrace{\Phi(\vec{r}')}_{V(x',y')} \frac{\partial G_D}{\partial n'}\,dS'$$

where $\nabla'^2 G_D(\vec{r},\vec{r}') = -4\pi\,\delta(\vec{r}-\vec{r}')$

$$G_D(\vec{r},\vec{r}')\Big|_S = 0$$

Then the solution we found above must be equivalent to

$$\Phi(\vec{r}) = -\frac{1}{4\pi}\int_{S(z=0)} V(x',y')\,\frac{\partial G_D}{\partial n'}\,dS'$$

but need to solve for G_D. To do this we need an orthonormal series expansion for G_D!

Example: Suppose we put a point charge q inside a grounded rectangular box

ie $\Phi(\vec{r})\Big|_S = 0$ (on all six faces)

Let $\vec{r}_0 = (x_0, y_0, z_0)$ be the position of the charge. To find the electrostatic potential for this configuration we must solve

$$\nabla^2 \Phi(\vec{r}) = -\frac{1}{\epsilon_0} q\, \delta(\vec{r}-\vec{r}_0) \quad \text{with } \Phi\Big|_S = 0$$

Consider a Fourier series expansion for Φ

$$\boxed{\Phi(\vec{r}) = \sum_{m,n,l=1}^{\infty} A_{m,n,l} \sqrt{\frac{8}{abc}} \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b} \sin\frac{l\pi z}{c}}$$

$A_{m,n,l}$ must be st $\nabla^2 \Phi(\vec{r}) = -\frac{q}{\epsilon_0} \underbrace{\delta(\vec{r}-\vec{r}_0)}_{\delta(x-x_0)\cdot\delta(y-y_0)\cdot\delta(z-z_0)}$

$$\nabla^2 \Phi(\vec{r}) = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial \Phi^2}{\partial z^2}$$

$$= \sum_{m,n,l=1} A_{m,n,l} \sqrt{\frac{8}{abc}} \left\{ -\left(\frac{m\pi x}{a}\right)^2 - \left(\frac{n\pi y}{b}\right)^2 - \left(\frac{l\pi z}{c}\right)^2 \right\} \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b} \sin\frac{l\pi z}{c}$$

For the RHS of Poisson's eq, recall the completeness condition

$$\sum_{n=1}^{\infty} u_n^*(x') u_n(x) = \delta(x'-x)$$

For our example we have

$$\frac{2}{a} \sum_{n=1}^{\infty} \sin\frac{m\pi x'}{a} \sin\left(\frac{m\pi x}{a}\right) = \delta(x'-x)$$

$$\vdots \qquad = \delta(y'-y)$$

$$= \delta(z'-z)$$

We represent $\delta(\vec{r}-\vec{r}_0)$ by

$$\delta(\vec{r}-\vec{r}_0) = \frac{8}{abc}\sum_{m,n,l}\left(\sin\frac{m\pi x_0}{a}\sin\frac{n\pi y_0}{b}\sin\frac{l\pi z_0}{c}\right)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sin\frac{l\pi z}{c}$$

Then:

$$-\sum_{m,n,l=1} A_{m,n,l}\sqrt{\frac{8}{abc}}\left\{\left(\frac{m\pi x}{a}\right)^2+\left(\frac{n\pi y}{b}\right)^2+\left(\frac{l\pi z}{c}\right)^2\right\}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sin\frac{l\pi z}{c}$$

$$= -\frac{1}{\epsilon_0}\frac{8}{abc}\sum_{m,n,l}\left(\sin\frac{m\pi x_0}{a}\sin\frac{n\pi y_0}{b}\sin\frac{l\pi z_0}{c}\right)$$

$$\times \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sin\frac{l\pi z}{c}$$

Identify recalls $A_{m,n,l}$

$$\therefore A_{m,n,l} = \frac{q}{\epsilon_0}\sqrt{\frac{8}{abc}}\frac{1}{\left(\frac{m\pi x}{a}\right)^2+\left(\frac{n\pi y}{b}\right)^2+\left(\frac{l\pi z}{c}\right)^2}\sin\frac{m\pi x_0}{a}\sin\frac{n\pi y_0}{b}\sin\frac{l\pi z_0}{c}$$

Can one solve this by another method ?!

$\boxed{2}$ Boundary value problems in electrostatics