ELECTROMAGNETISM (PART B)

Chapter 2: Boundary value problems in electrostatics (Part 4)

Lecture 7

$\boxed{2}$ Boundary value problems in electrostatics

$\boxed{2.1}$ Boundary value problems

$\boxed{2.2}$ Green's functions

$\boxed{2.3}$ Method of images

$\boxed{2.4}$ Method of orthonormal functions

$\boxed{2.4.1}$ Generalities

$\boxed{2.4.2}$ Electrostatic problems with rectangular symmetry

this lecture → $\boxed{2.4.3}$ Green's functions

$\boxed{2.4.4}$ Spherical symmetries

2.4.4 Green's functions: expansion in orthonormal functions

Example: Green's function for Dirichlet boundary condition on the plane $z=0$

ie $\quad \nabla'^2 G(\vec{r},\vec{r}') = -4\pi\,\delta(\vec{r}-\vec{r}')$ with $G(\vec{r},\vec{r}')\Big|_{z'=0} = 0$

we did this already

Recall: $\quad G_D(\vec{r},\vec{r}') = \dfrac{1}{|\vec{r}-\vec{r}'|} - \dfrac{1}{|\vec{r}-\vec{r}_R'|}$ $\qquad \vec{r}_R = (x, y, -z)$

is a solution to this problem

Note: $G_D(\vec{r},\vec{r}') = G_D(\vec{r}',\vec{r})$ so $G(\vec{r},\vec{r}')\big|_{z'=0} = 0$

Consider the set of orthonormal functions

$$\frac{1}{\sqrt{2\pi}} e^{i\alpha x}, \quad \frac{1}{\sqrt{2\pi}} e^{i\beta y}, \quad \frac{1}{\sqrt{2\pi}} e^{i\gamma z}$$

Using the completeness relation, we can represent $\delta(\vec{r}-\vec{r}\,')$ in terms of these functions by

$$\delta(\vec{r}-\vec{r}\,') = \delta(x-x')\,\delta(y-y')\,\delta(z-z')$$

$$= \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} d\gamma\, e^{i\alpha(x-x')} e^{i\beta(y-y')} e^{i\gamma(z-z')}$$

closure: $\delta(x-x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(x-x')k} \, dk$

For the Green's function we have

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} d\gamma \underbrace{A(\alpha,\beta,\gamma)}_{\text{Fourier coeffs}} e^{i\alpha(x-x')} e^{i\beta(y-y')} e^{i\gamma(z-z')}$$

st G satisfies Green's equation

Then

$$\nabla^2 \frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} d\gamma \, A(\alpha,\beta,\gamma)(-\alpha^2-\beta^2-\gamma^2) \times e^{i\alpha(x-x')} e^{i\beta(y-y')} e^{i\gamma(z-z')}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \cdots$$

$$= -4\pi\,\delta(\vec{r}-\vec{r}')$$

Fourier integral

Then $$\nabla'^2 \frac{1}{|\vec{r}-\vec{r}'|} = -4\pi\,\delta(\vec{r}-\vec{r}')$$

gives the Fourier coefficients $A(\alpha, \beta, \gamma)$.

$$-\frac{1}{(2\pi)^{3/2}}(\alpha^2+\beta^2+\gamma^2)\,A(\alpha,\beta,\gamma) = -4\pi \times \frac{1}{(2\pi)^3}$$

$$\Rightarrow \quad A(\alpha,\beta,\gamma) = \frac{4\pi}{(2\pi)^{3/2}} \quad \frac{1}{(\alpha^2+\beta^2+\gamma^2)}$$

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} d\gamma\, A(\alpha,\beta,\gamma)\, e^{i\alpha(x-x')}\, e^{i\beta(y-y')}\, e^{i\gamma(z-z')}$$

$$= \frac{1}{2\pi^2}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} d\gamma\, \frac{e^{i\alpha(x-x')}\, e^{i\beta(y-y')}\, e^{i\gamma(z-z')}}{(\alpha^2+\beta^2+\gamma^2)}$$

We thus obtain a Fourier integral for $\frac{1}{|\vec{r}-\vec{r}\,'|}$

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{2\pi^2}\int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\beta \boxed{\int_{-\infty}^{\infty} d\gamma \, \frac{e^{i\alpha(x-x')}\, e^{i\beta(y-y')}\, e^{i\gamma(z-z')}}{(\alpha^2+\beta^2+\gamma^2)}}$$

Integrating over γ (proof in a moment)

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{2\pi^2}\int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\beta \, e^{i\alpha(x-x')}\, e^{i\beta(y-y')} \frac{\pi e^{-\sqrt{\alpha^2+\beta^2}\,|z-z'|}}{\sqrt{\alpha^2+\beta^2}}$$

Finally: the Fourier integral for the Dirichlet Green's function is

$$G_D(\vec r, \vec r') = \frac{1}{|\vec r - \vec r'|} - \frac{1}{|\vec r - \vec r_R'|} \qquad \vec r_R' = (x', y', -z')$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\beta \, \frac{e^{i\alpha(x-x')} e^{i\beta(y-y')}}{\sqrt{\alpha^2+\beta^2}} \left\{ e^{-|z-z'|\sqrt{\alpha^2+\beta^2}} - e^{-(z+z')\sqrt{\alpha^2+\beta^2}} \right\}$$

- $G_D(\vec r, \vec r') = 0$ for $z=0$ or $z'=0$ ✓

Integral over γ: prove that

$$I = \int_{-\infty}^{\infty} d\gamma \, \frac{e^{i\gamma(z-z')}}{(\alpha^2+\beta^2+\gamma^2)} = \pi \, \frac{e^{-\sqrt{\alpha^2+\beta^2}\,|z-z'|}}{\sqrt{\alpha^2+\beta^2}}$$

Sketch: let $Z = z - z'$ $\qquad A^2 = \alpha^2+\beta^2$

want to prove $\quad I = \int_{-\infty}^{\infty} d\gamma \, \frac{e^{i\gamma z}}{(A^2+\gamma^2)} = \pi \, \frac{e^{-Az}}{A}$

$$I = \int_{-\infty}^{\infty} d\gamma \, e^{i\gamma z} \, \frac{1}{2A}\left(\frac{1}{A+i\gamma} + \frac{1}{A-i\gamma} \right) = I_1 + I_2$$

Strategy: rewrite this integral as a contour integral over the complex plane & then use the residue thm.

let's compute : $I_1 = \frac{1}{2A}\int_{-\infty}^{\infty} d\gamma \, \frac{e^{i\gamma z}}{A+i\gamma}$

Set $w = (A+i\gamma) z$

$\Rightarrow dw = iz\, d\gamma$, $\frac{dw}{w} = i \frac{1}{A+i\gamma} d\gamma$

$\boxed{\text{For } z>0}$, $-\infty < \gamma < \infty \rightsquigarrow -\infty < \underset{\gamma z}{\overset{\shortparallel}{\operatorname{Im} w}} < \infty$

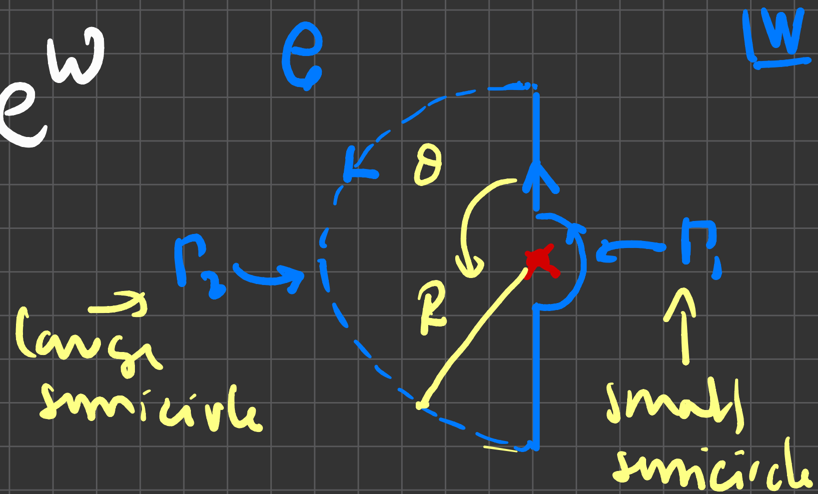
$z - z' > 0$

$$I_1 = \frac{1}{2A} \int_{-\infty < \operatorname{Im} w < \infty} (-i)\, e^{w - Az} \frac{dw}{w}$$

$$= -\frac{i}{2A} e^{-Az} \int_{-\infty < \operatorname{Im} w < \infty} dw \, \frac{e^w}{w}$$

simple pole at $w = 0$

Claim: $$I_1 = -\frac{i}{2A}\, e^{-Az} \oint_{\mathcal{C}} \frac{dw}{w}\, e^{w}$$



Integral over Γ_2 (large semicircle):

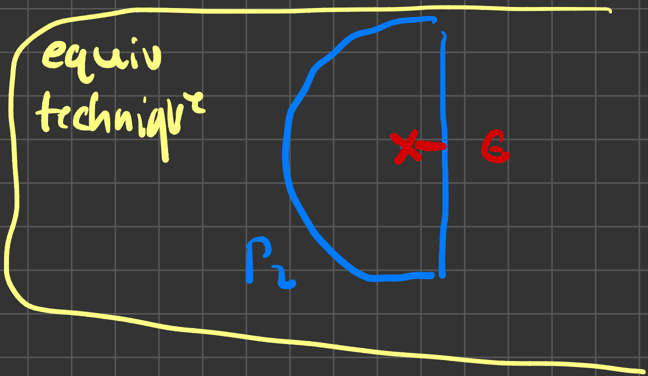
$$w = R\, e^{i\theta}, \quad \frac{\pi}{2} < \theta < \frac{3\pi}{2} \qquad \frac{dw}{w} = \frac{iRe^{i\theta}\, d\theta}{Re^{i\theta}} = i\, d\theta$$

$R = |w|$

$$\int_{\Gamma_2} \rightsquigarrow i \int_{\theta=\pi/2}^{\theta=3\pi/2} d\theta\, e^{R(\cos\theta + i\sin\theta)}$$

$-1 \le \cos\theta \le 0$

$\longrightarrow 0$ as $R \longrightarrow \infty$



Similar computation for $\int_{\Gamma_1}$, Γ_1 small semicircle: $\int_{\Gamma_1} \cdots \to 0$

$$I_1 = \frac{1}{2A}\int_{-\infty}^{\infty} d\gamma \, \frac{e^{i\gamma z}}{A + i\gamma} = -\frac{i}{2A} e^{-Az} \oint_C \frac{dw}{w} e^{w}$$

simple pole at $w=0$

Now use the residue theorem

$$I_1 = -\frac{i}{\cancel{2}A} e^{-Az} \cdot \cancel{2}\pi i \underbrace{(\text{residue of the pole at } w=0)}_{\lim\limits_{w\to c}(w-c)f(w) = \lim\limits_{w\to 0} e^{w} = 1}$$

$c = 0$, $f(w) = \frac{e^w}{w}$

$$= \frac{\pi}{A} e^{-Az} \cdot 1$$

Finally: $I_1 = \frac{\pi}{\sqrt{\alpha^2+\beta^2}} e^{-\sqrt{\alpha^2+\beta^2}\,(z-z')}$ for $z > z'$

$(z = z - z' \qquad A^2 = \alpha^2 + \beta^2)$

Next: problems with spherical symmetry