ELECTROMAGNETISM (PART B)

Chapter 2: Boundary value problems in electrostatics (Part 5)

Lecture 8

2 Boundary value problems in electrostatics

2.1 Boundary value problems

2.2 Green's functions

2.3 Method of images

2.4 Method of orthonormal functions

2.4.1 Generalities

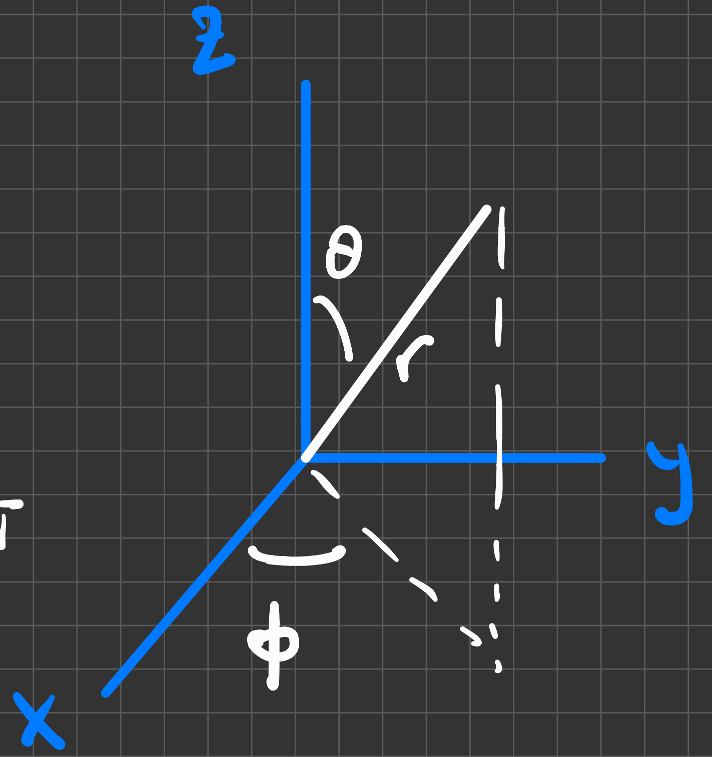
2.4.2 Electrostatic problems with rectangular symmetry

2.4.3 Green's functions

this lecture

2.4.4 Spherical symmetries

2.4.4 Spherical symmetries



spherical coordinates (r, θ, ϕ)

$0 \leq \theta < \pi \;;\; 0 \leq \phi < 2\pi$

Laplacian:

$$\nabla^2 \Phi = \frac{1}{r} \frac{\partial^2}{\partial r^2}(r\Phi) + \underbrace{\frac{1}{r^2} \frac{1}{\sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2} \frac{1}{\sin^2\theta} \frac{\partial^2\Phi}{\partial\phi^2}}_{\frac{1}{r^2} \nabla^2_{\theta,\phi} \Phi}$$

Consider separable solutions of $\nabla^2 \Phi = 0$

$$\Phi(r, \theta, \phi) = R(r)\, Y(\theta, \phi)$$

Then

$$\nabla^2 \Phi = Y(\theta, \phi) \frac{1}{r} \frac{d^2}{dr^2}(r R(r)) + \frac{1}{r^2} R(r)\, \nabla^2_{\theta, \phi} Y(\theta, \phi)$$

$\therefore$ $\times \dfrac{r^2}{RY}$

$$\underbrace{\frac{r}{R} \frac{d^2}{dr^2}(r R(r))}_{c} + \underbrace{\frac{1}{Y(\theta, \phi)} \nabla^2_{\theta, \phi} Y(\theta, \phi)}_{-c} = 0$$

$c \in \mathbb{R}$ a constant

$\therefore$ $\dfrac{d^2}{dr^2}(rR) = c\dfrac{R}{r}$, $\nabla^2_{\theta, \phi} Y(\theta, \phi) = -c\, Y(\theta, \phi)$

Let $Y(\theta, \phi) = P(\theta)Q(\phi)$ spherical harmonics

$$\nabla^2_{\theta,\phi} Y(\theta, \phi) = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial \Phi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 \Phi}{\partial\phi^2}$$

$$= Q(\phi)\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d}{d\theta}P(\theta)\right) + \frac{1}{\sin^2\theta}P(\theta)\frac{d^2Q(\phi)}{d\phi^2}$$

$$= -C\, P(\theta)Q(\theta) \qquad \times \frac{\sin^2\theta}{PQ}$$

$$\underbrace{\frac{\sin\theta}{P(\theta)}\frac{d}{d\theta}\left(\sin\theta \frac{d}{d\theta}P(\theta)\right) + C\sin^2\theta}_{\lambda} + \underbrace{\frac{1}{Q}\frac{d^2Q(\phi)}{d\phi^2}}_{-\lambda} = 0$$

$$\frac{d^2 \Phi}{d\phi^2} + \lambda \Phi = 0$$

Want Φ to be periodic and single valued under

$$\phi \longrightarrow \phi + 2\pi$$

That is :

$$\Phi(\phi) = \Phi(\phi + 2\pi)$$

This means that $\lambda = + m^2$ $\quad m \in \mathbb{Z}$

and so

$$\Phi(\phi) = e^{\pm i m \phi}$$

For $P(\theta)$:

$$\frac{\sin\theta}{P(\theta)}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}P(\theta)\right)+c\sin^2\theta = m^2 \qquad \frac{P(\theta)}{\sin^2\theta}$$

$$\Rightarrow \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}P(\theta)\right)+\left(\ell(\ell+1)-\frac{m^2}{\sin^2\theta}\right)P(\theta)=0$$

where we have set $c=\ell(\ell+1)$, $\ell\in\mathbb{R}$

Set $x = \cos\theta$ $\qquad \frac{d}{d\theta} = -\sin\theta \frac{d}{dx}$

$$\frac{d}{dx}\left((1-x^2)\frac{dP}{dx}\right) + \left(\ell(\ell+1) - \frac{m^2}{1-x^2}\right)P = 0$$ Legendre equation

solutions : $P_{\ell,m}(x) \rightsquigarrow$ Legendre functions

(case $m^2 = 0$: ordinary differential eq with
solns $\rightsquigarrow$ Legendre polynomials)

Solutions should be single valued, finite and continuous on $-1 \le x \le 1$ (including the N & S poles)

$\theta = 0, \pi$

$\therefore \quad \ell \in \mathbb{Z} \quad \& \quad |m| \le \ell$

Radial equation: $\frac{d^2}{dr^2}(rR) = l(l+1)\frac{R}{r}$

Let $R = r^{\alpha}$, $\alpha \neq 0$

Then $\alpha(\alpha+1) r^{\alpha-1} = l(l+1) r^{\alpha-1} \Rightarrow \alpha(\alpha+1) = l(l+1)$

$\Rightarrow \alpha = l, -(l+1)$

$\Rightarrow R = \underline{r^{l}, r^{-(l+1)}}$

Finally:

spherical harmonics

$$\Phi(\vec{r}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(A_{\ell,m} r^{\ell} + B_{\ell m} r^{-(\ell+1)}\right) Y_{\ell m}(\theta, \phi)$$

$$P_{\ell,m}(\theta)\, e^{\pm i m \phi}$$

Legendre functions

$$\nabla^2 Y_{\ell,m} = -\ell(\ell+1) Y_{\ell,m}$$

where

$$\frac{d}{dx}\left((1-x^2)\frac{dP}{dx}\right) + \left(\ell(\ell+1) - \frac{m^2}{1-x^2}\right)P = 0$$

Legendre eq

Spherical harmonics form a complete set of orthonormal functions.

One can prove

① $$Y_{\ell,-m}(\theta,\phi) = (-1)^m \, Y^*_{\ell,m}(\theta,\phi)$$

② $$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \; Y^*_{\ell',m'}(\theta,\phi) \, Y_{\ell,m}(\theta,\phi) = \delta_{\ell\ell'}\,\delta_{mm'}$$

(orthonormality condition)

③ $$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y^*_{\ell,m}(\theta',\phi')\, Y_{\ell,m}(\theta,\phi) = \delta(\phi-\phi')\, \underbrace{\delta(\cos\theta-\cos\theta')}_{\text{not } \delta(\theta-\theta')!}$$

(closure)

First few $\quad l \in \mathbb{Z} \quad \& \quad |m| \leq l$

$l=0 \quad m=0 \qquad Y_{00} = \frac{1}{\sqrt{4\pi}}$

$l=1 \quad m=0 \qquad Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta$

$m=1 \qquad Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi}$

$l=2 \quad m=0 \qquad Y_{20} = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2}\cos^2\theta - \frac{1}{2} \right)$

$m=1 \qquad Y_{21} = -\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta \, e^{i\phi}$

$m=2 \qquad Y_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \sin^2\theta \, e^{2i\phi}$

Example: compute Φ $\boxed{\text{inside}}$ a sphere of radius a with

$$\Phi(a,\theta,\phi) = V(\theta,\phi) \quad \text{prescribed}$$

$$\Phi(r,\theta,\phi) \text{ bounded as } r \to 0$$

consider a solution of the form

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{l,m}\, r^l\, Y_{l,m}(\theta,\phi)$$

no negative powers of r
why?

why? no neg powers of r otherwise Φ would not be bounded at $r=0$

Imposing the boundary condition at $r = a$ specifies the coeffs A_{lm}:

$$\Phi(a,\theta,\phi) = V(\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a^{l} A_{l,m} Y_{l,m}(\theta,\phi)$$

series for V in spherical harmonics

Using the orthonormality condition to extract the coeffients

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \, \sin\theta \, V(\theta,\phi) \, Y^*_{lm}(\theta,\phi)$$

$$= \sum_{l'=0}^{\infty} \sum_{m'=-l'}^{l'} a^{l'} A_{l',m'} \underbrace{\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \, \sin\theta \, Y_{l,m}(\theta,\phi) Y^*_{l',m'}(\theta,\phi)}_{\delta_{l,l'} \delta_{m,m'}}$$

$$= a^{l} A_{l,m}$$

Green's functions: Fourier expansion

Determine Green's functions for problems in electrostatics involving (spherical) charge distributions and boundary conditions for Φ in terms of an expansion of spherical harmonics

Example: determine the Dirichlet Green's function for the $\boxed{\text{exterior}}$ of a sphere of radius a

in terms of spherical ON functions

Let
$$G(\vec{r}, \vec{r}\,') = \sum_{l,m} \tilde{A}_{l,m}(r, \vec{r}\,')\, Y_{l,m}(\vartheta, \phi)$$

st
$$\nabla'^2 G(\vec{r}, \vec{r}\,') = -4\pi\, \delta(\vec{r} - \vec{r}\,')$$
$$G(\vec{r}, \vec{r}\,')\big|_{r=a} = 0 \quad , \quad (G(\vec{r}, \vec{r}\,') = G(\vec{r}\,', \vec{r}))$$

Then

$$\nabla^2 G(\vec{r},\vec{r}') = \sum_{l,m} \left\{ \frac{1}{r} \frac{d^2}{dr^2} (r\tilde{A}(r,\vec{r}')) - \frac{1}{r^2} l(l+1) \tilde{A}(r,\vec{r}') \right\} Y_{l,m}(\theta,\phi)$$

$$\frac{1}{r^2} \nabla^2_{\theta,\phi} Y_{l,m}(\theta,\phi)$$

$$= -\frac{1}{r^2} l(l+1) Y_{lm}$$

We need $\delta(\vec{r}-\vec{r}')$ in spherical coordinates

Claim: in spherical coordinates

$$\delta(\vec{r}-\vec{r}') = \frac{1}{r^2}\,\delta(r-r')\,\delta(\phi-\phi')\,\underbrace{\delta(\cos\theta-\cos\theta')}_{\text{not } \delta(\theta-\theta')}$$

In Cartesian coordinates: $\delta(\vec{r}-\vec{r}') = \delta(x-x')\,\delta(y-y')\,\delta(z-z')$

Now note that naively

non sense!
$$\int_V \underbrace{\delta(r)\,\delta(\theta)\,\delta(\phi)}_{\delta(\vec{r}-\vec{r}')}\;\underbrace{r^2 \sin\theta\, dr\, d\theta\, d\phi}_{dV} \neq \begin{cases} 1 & \text{origin} \in V \\ 0 & \text{"} \notin V \end{cases}$$

Instead $\delta(\vec{r}-\vec{r}')\,dV'$ is invariant under coordinate transformations, in particular under $(x,y,z) \longrightarrow (r,\theta,\phi)$

$$\int_V \delta_{cc}(\vec{r}-\vec{r}')\,dV' = \int_V \underbrace{\delta_{sc}(\vec{r}-\hat{r}')}\ \underbrace{\overbrace{r'^2 dr' \sin\theta' d\theta' d\phi'}^{\text{Jac of transformation}}}_{dV' \text{ in spherical coords}}$$

Then
$$\delta_{sc}(\vec{r}-\hat{r}') = \frac{1}{r'^2 \sin\theta'}\,\delta(r-r')\,\delta(\theta-\theta')\,\delta(\phi-\phi')$$
$$= \frac{1}{r^2}\,\delta(r-r')\,\delta(\cos\theta-\cos\theta')\,\delta(\phi-\phi')$$

last step by property of dirac δ-function

$$\delta(f(x)) = \frac{1}{|f(x)'|}\,\delta(x-x_0) \qquad f(x_0)=0$$

function with a simple zero at x_0

Then

$$\sum_{l,m}\left\{\frac{1}{r}\frac{d^2}{dr^2}\left(r\tilde{A}(r,\vec{r}')\right)-\frac{1}{r^2}l(l+1)\tilde{A}(r,\vec{r}')\right\}Y_{l,m}(\theta,\phi) = \nabla^2 G(\vec{r},\vec{r}')$$

$$\parallel$$

$$= -4\pi\frac{1}{r^2}\delta(r-r')\underbrace{\delta(\cos\theta-\cos\theta')\,\delta(\phi-\phi')}_{\text{closure relation}} = -4\pi\,\delta(\vec{r}-\vec{r}')$$

$$= -\frac{4\pi}{r^2}\delta(r-r')\sum_{l,m}Y^*_{l,m}(\theta',\phi')\,Y_{l,m}(\theta,\phi)$$

This gives

$$\frac{1}{r}\frac{d^2}{dr^2}\left(r\tilde{A}(r,\vec{r}')\right)-\frac{1}{r^2}l(l+1)\tilde{A}(r,\vec{r}') = -\frac{4\pi}{r^2}\delta(r-r')\,Y^*_{l,m}(\theta',\phi')$$

This means:

$$\underline{\tilde{A}(r,\vec{r}') = A(r,r')\,Y^*_{lm}(\theta',\phi')}$$

$$\Rightarrow G(\vec{r}, \vec{r}') = \sum_{l,m} \tilde{A}_{l,m}(r, \vec{r}')\, Y_{l,m}(\theta, \phi)$$

$$= \sum_{l,m} A_{l,m}(r, r')\, Y^*_{l,m}(\theta', \phi')\, Y_{l,m}(\theta, \phi)$$

where $A_{l,m}(r, r')$ satisfies the radial equation

$$\frac{1}{r} \frac{d^2}{dr^2}\left(r A_{l,m}(r, r')\right) - \frac{1}{r^2} l(l+1) A_{lm}(r, r') = -\frac{4\pi}{r^2} \delta(r - r')$$

(A_{lm} is independent of m)

Solving the radial eq for $A_{\ell,m}(r,r')$:

First, let $\boxed{r \neq r'}$ so $r > r'$ or $r < r'$

Then

$$\frac{d^2}{dr^2}(r A_{\ell,m}(r,r')) = \frac{1}{r}\ell(\ell+1) A_{\ell,m}(r,r')$$

Solutions: $r^\ell, \quad r^{-(\ell+1)}$

$$\underline{A_{\ell,m}}(r,r') = \begin{cases} A(r')r^\ell + B(r')r^{-(\ell+1)} & a < r < r' \\ \cancel{C(r')}r^\ell + D(r')r^{-(\ell+1)} & r > r' > a \end{cases}$$

$\to 0$

$C(r') = 0$ otherwise as $r \to \infty$ $A \to \infty$

want $G(\vec{r},\vec{r}')$ bounded as $r \to \infty$

To determine A, B, D we

1. Impose Dirichlet boundary conditions

$$G_0 = 0 \quad \text{at} \quad r = a$$

2. symmetry $\quad r \leftrightarrow r'$

3. behaviour at $r = r'$

1. Dirichlet boundary conditions at $r=a$

$$G_D(\vec{r},\vec{r}')\Big|_{r=a} = 0$$

$a \leq r < r'$ that is: $A_{\ell m}(a, r') = A(r')a^{\ell} + B(r')a^{-(\ell+1)} = 0$

$$\Longrightarrow \qquad B(r') = -a^{2\ell+1} A(r')$$

$$A_{\ell m}(r, r') = \begin{cases} A(r')\left(r^{\ell} - \dfrac{a^{2\ell+1}}{r^{\ell+1}}\right), & a < r < r' \quad (a) \\[2ex] D(r')\dfrac{1}{r^{\ell+1}}, & r > r' > a \quad (b) \end{cases}$$

② symmetry $r \leftrightarrow r'$: $A_{l,m}(r,r') = A_{l,m}(r',r)$

$$A_{l,m}(r',r) = \begin{cases} A(r)\left(r'^l - \frac{a^{2l+1}}{r'^{l+1}}\right), & r > r' > a \quad (a)' \\ D(r)\frac{1}{r'^{l+1}}, & a < r < r' \quad (b)' \end{cases}$$

$(b) = (a)'$ $$D(r') = \left(r'^l - \frac{a^{2l+1}}{r'^{l+1}}\right) r^{l+1} A(r) \quad (*)$$

$(b)' = (a)$ gives $(*)$ with $r \leftrightarrow r'$

$$(*) \Leftrightarrow \underbrace{D(r')\left(r'^l - \frac{a^{2l+1}}{r'^{l+1}}\right)^{-1}}_{\text{only } r'} = \underbrace{A(r)\, r^{l+1}}_{\text{only } r} = \underset{\substack{\uparrow \\ \text{constant}}}{K}$$

$\therefore \quad A(r) = \dfrac{K}{r^{\ell+1}}$ and so $A(r') = \dfrac{K}{r'^{\ell+1}}$

and $\quad D(r') = K r'^{\ell} \left(1 - \dfrac{a^{2\ell+1}}{r'^{(2\ell+1)}} \right)$

$$A_{\ell,m}(r', r) = \begin{cases} K \dfrac{r^{\ell}}{r'^{\ell+1}} \left(1 - \dfrac{a^{2\ell+1}}{r^{2\ell+1}} \right), & a < r < r' \\ K \dfrac{r'^{\ell}}{r^{\ell+1}} \left(1 - \dfrac{a^{2\ell+1}}{r'^{(2\ell+1)}} \right), & r > r' > a \end{cases}$$

3 $\underline{r = r'}$

$$\frac{d^2}{dr^2}\left(r A_{\ell,m}\right) - \frac{1}{r}\ell(\ell+1) A_{\ell,m} = -\frac{4\pi}{r}\delta(r-r')$$

Integrate this equation over r from $r = r' - \epsilon$ to $r = r' + \epsilon$ and then let $\epsilon \to 0$

$$-\frac{4\pi}{r} = \frac{d}{dr}\left(r A_{\ell,m}\right)\Big|_{r=r'+\epsilon} - \frac{d}{dr}\left(r A_{\ell m}\right)\Big|_{r=r'-\epsilon}$$

A for $r > r'$

A for $r < r'$

$$\cancel{-\ell(\ell+1)\int_{r=r'-\epsilon}^{r=r'+\epsilon}\frac{1}{r}A_{\ell m}\,dr} \to 0$$

exercise

$$\int_{r=r'-\epsilon}^{r=r'+\epsilon} \cdots = \int_{r=r'-\epsilon}^{r} \cdots + \int_{r}^{r'+\epsilon} \cdots$$

$$-\frac{4\pi}{r} = K \left. \frac{d}{dr}\left(\frac{r'^{\ell}}{r^{\ell}}\left(1-\frac{a^{2\ell+1}}{r'^{(2\ell+1)}}\right)\right)\right|_{r=r'+\epsilon} - \left. \frac{d}{dr}\left(\frac{r^{\ell+1}}{r'^{\ell+1}}\left(1-\frac{a^{2\ell+1}}{r^{2\ell+1}}\right)\right)\right|_{r=r'-\epsilon}$$

$$= K\left\{-\ell\,\frac{(r-\epsilon)^{\ell}}{r^{\ell+1}}\left(1-\frac{a^{2\ell+1}}{(r-\epsilon)^{2\ell+1}}\right) - \frac{1}{(r+\epsilon)^{\ell+1}}\left((\ell+1)r^{\ell}+\ell a^{2\ell+1}\frac{1}{r^{\ell+1}}\right)\right\}$$

$$\overset{\epsilon=0}{=} K\left\{-\frac{\ell}{r}\left(1-\cancel{\frac{a^{2\ell+1}}{r^{2\ell+1}}}\right)-\frac{(\ell+1)}{r} - \cancel{\ell\frac{a^{2\ell+1}}{r^{2\ell+2}}}\right\} = -K(2\ell+1)\frac{1}{r}$$

$$\Longrightarrow \quad K = \frac{4\pi}{2\ell+1}$$

Finally:

$$G(\vec{r}, \vec{r}\,') = \sum_{l,m} A_{l,m}(r, r')\, Y^*_{l,m}(\theta', \phi')\, Y_{lm}(\vartheta, \phi)$$

where $$A_{lm}(r, r') = \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} \left(1 - \frac{a^{2l+1}}{r_<^{2l+1}}\right)$$

$$r_> = r' \qquad r_< = r \qquad r < r'$$
$$r_> = r \qquad r_< = r' \qquad r > r'$$

Note that indeed $G(\vec{r}, \vec{r}\,')\Big|_{r=a} = 0$, also at $r' = a$

//

Next: Magnetostatics