B7.2 ELECTROMAGNETISM

Chapter 3: Magnetostatics (Part 1)

Lecture 9

$\boxed{3}$ Magnetostatics

Magnetic fields are produced by electric currents

Magnetostatics: study of magnetic phenomena due to time independent currents

As we will see in this chapter: electric & magnetic phenomena seem different. For example: there are $\boxed{\text{no}}$ point like magnetic charges

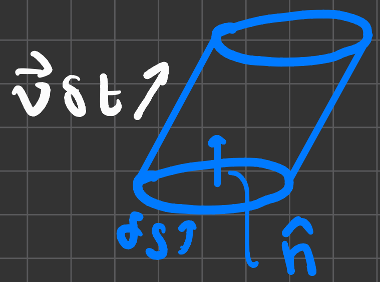
$\boxed{3.1}$ Currents & conservation of charge

Charges in motion are described by a current density

$$\vec{J}(t, \vec{r})$$ vector field.

Defining $\vec{J}(t,\vec{r})$:

Suppose we have a distribution of charge density $\rho(t,\vec{r})$ crossing a surface δS with velocity $\vec{v}$
(δS is chosen sufficiently small so that $\vec{v}$ is constant across δS)
Then a small region δV with cross section δS and length $\vec{v}\,\delta t$ contains a charge



$$\rho\,\delta V = \rho\,\underbrace{\vec{v}\,\delta t\cdot\delta\vec{S}}_{\delta V} \equiv \vec{J}\cdot\delta\vec{S}\,\delta t$$

total charge passing through δS in the time δt

so $\vec{J}$ is the vector field in the direction of the flow and its magnitude is the amount of charge crossing a unit area per unit time.

We also define the electric current through a surface S

$$I(S) = \int_S \vec{J} \cdot d\vec{S}$$

rate of flow of charge through S

units

$$[I] = \text{Ampère } (A)$$

$$= \text{coul sec}^{-1}$$

Conservation of charge

* electric charge is conserved ✓ continuity equation that expresses this experimental fact.

Consider a closed surface $S = \partial V$ bounding a region V where there are charges moving in & out through S

Let Q be the total charge inside V : $Q = \int_V \rho \, dV$

- rate of change of Q (decrease of the amount of charge in V) $= -\frac{dQ}{dt} = -\frac{d}{dt}\int_V \rho \, dV = -\int_V \frac{\partial \rho}{\partial t} \, dV$

$\parallel$

flow of charge out through $S = \partial V$ (current through S) $= \int_S \vec{J} \cdot d\vec{S} = \int_V \nabla \cdot \vec{J} \, dV$

$$\Longrightarrow \quad \int_S \left(\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} \right) dV = 0 \quad \forall \text{ regions } V \subset \mathbb{R}^3$$

Then:

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0}$$

charge conservation law (continuity eq)

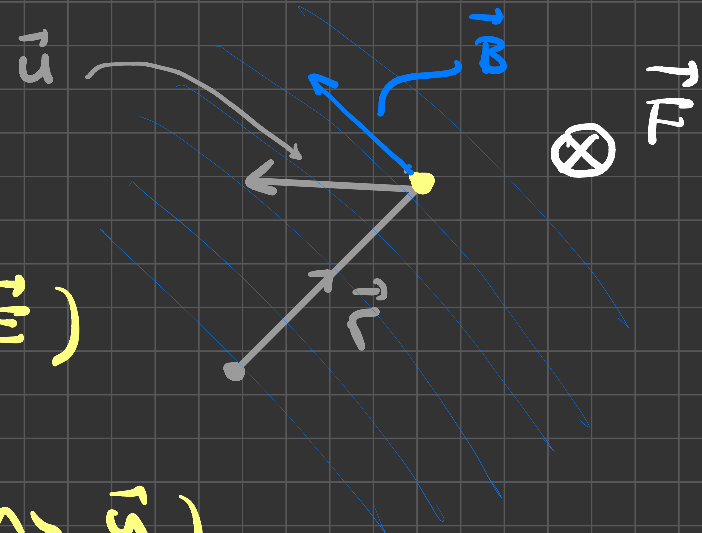
Magnetostatics $\rightarrow$ stationary distributions so $\frac{\partial \rho}{\partial t} = 0$

$$\Longrightarrow \quad \underline{\nabla \cdot \vec{J} = 0}$$

$\boxed{3.2}$ Biot-Savart law & the magnetic field strength

A test particle with charge e moving with velocity $\vec{u}$ in a magnetic field $\vec{B}$ is subject to a force

Lorentz force law $$\vec{F} = e\,\vec{u} \wedge \vec{B}$$



▷ $\vec{F}$ is perpendicular to $\vec{u}$ and $\vec{B}$

(compare with Coulomb Force law $\vec{F} \parallel \vec{E}$)

▷ $\vec{u} \cdot \vec{F} = 0$ ($\vec{F}$ has no component along $\vec{u}$)

This means the magnetic force does no work on e

[recall work done by a force $\vec{F}$ along a path connecting $\vec{r}_1$ & $\vec{r}_2$: $W = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r} = \int_{P_1}^{P_2} \vec{F} \cdot \vec{u}\, dt = 0$]

But what produces $\vec{B}$?

Biot-Savart law: (experimental)

- a current generates a magnetic field $\vec{B}$
  (law gives a relation between $\vec{B}$ & $\vec{J}$)

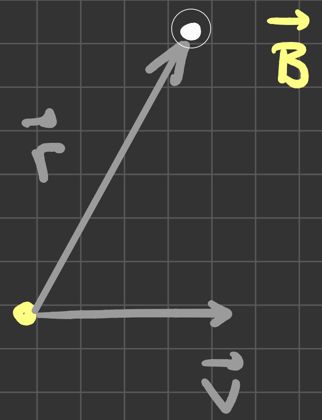
A charge q moving with velocity $\vec{v}$ at the origin produces a magnetic field $\vec{B}(\vec{r})$ at the point with position $\vec{r}$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{q\vec{v} \wedge \vec{r}}{r^2}$$

units: $[B] = \text{Tesla}$ (force per unit of current)

$\mu_0 = 4\pi \times 10^{-7}\ \text{Nt sec}^2\ \text{Coul}^{-2}$ permeability of the vacuum

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\,\frac{\overbrace{q\vec{v}}^{\vec{J}} \wedge \vec{r}}{r^2}$$



- $\vec{B}$ is perpendicular to $\vec{r}$ and $\vec{v}$

- $|\vec{B}| \propto \dfrac{1}{r^2}$

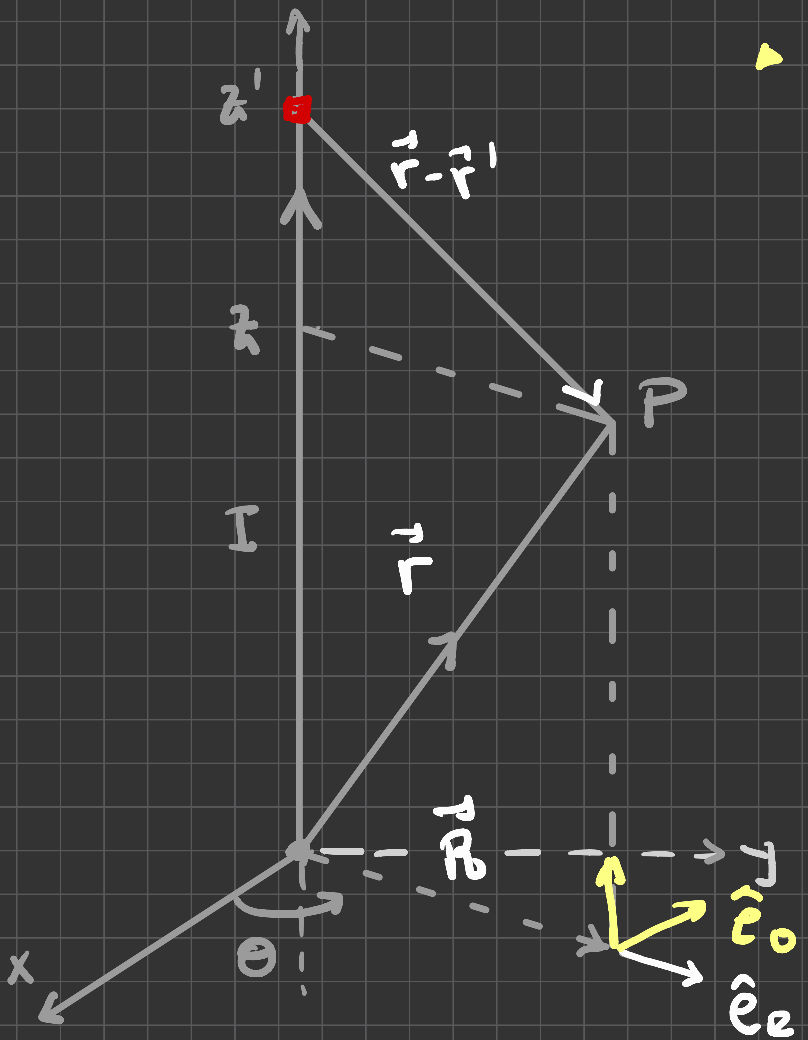
By the superposition principle, the magnetic field due to a current density $\vec{J}(\vec{r})$ in a region V is

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J}(\vec{r}\,')\wedge(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\,dV'$$

Example: Consider a straight infinite wire carrying a steady current

$$\vec{I} = I\,\hat{k}$$

Compute magnetic field generated by this current.



- use cylindrical coordinates (R, θ, z)

  so $\vec{r}$ be the position vector of the point of observation P

$$\vec{r} = \vec{R} + z\hat{k} = R\hat{e}_R + z\hat{k}$$

$$(\hat{e}_R = \cos\theta\,\hat{i} + \sin\theta\,\hat{j})$$

- The position $\vec{r}\,'$ of an element of current is

$$\vec{r}\,' = z'\,\hat{k}$$

- $$\vec{r} - \vec{r}\,' = \vec{R} + (z - z')\hat{k}$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int_{-\infty}^{\infty} \frac{I\,\hat{k}\,dz' \wedge (\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

$$\vec{J}(\vec{r}')\,dV' \to I\,\hat{k}\,dz'$$

$$\hat{k}\wedge|\vec{r}-\vec{r}'| = \hat{k}\wedge\left(\vec{R} + \cancel{(z-z')\hat{k}}\right)$$

$$= R\,\hat{k}\wedge\hat{e}_R = R\,\hat{e}_\theta$$

Then $\vec{B} = B\,\hat{e}_\theta$, $\quad B = \frac{\mu_0 I R}{4\pi}\int_{-\infty}^{\infty}\frac{1}{|R^2+(z-z')^2|^{3/2}}\,dz'$

$$= \frac{\mu_0 I}{2\pi R} = B(R)$$

In cartesian coordinates $\quad \vec{B} = \frac{\mu_0 I}{2\pi R}\frac{1}{R}(-y, x, 0)$

$$R = \sqrt{x^2+y^2}, \quad \hat{e}_\theta = \frac{1}{R}(-y, x, 0)$$

$$= (-\sin\theta, \cos\theta, 0)$$

We could have used symmetries to show $\vec{B} = B(R)\,\hat{e}_\theta$:

$\vec{B}$ must be perpendicular to $\hat{n}$ so a priori $\vec{B}$ has components along $\hat{e}_\theta$ and $\hat{e}_R$. $\vec{B}$ is also perpendicular to $\vec{r}-\vec{r}\,'$

so $(\vec{r}-\vec{r}\,')\cdot\vec{B} = \vec{R}\cdot\vec{B} = 0$ hence $\vec{B} = B\,\vec{e}_\theta$

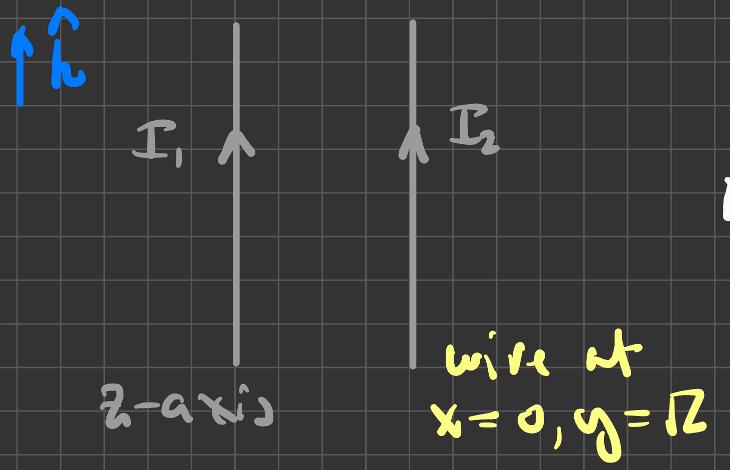
Also $|\vec{B}| = B(R)$ by cylindrical symmetry

What is the force at P on a test particle of charge e moving with velocity $\vec{u}$

$$\vec{F} = e\,\vec{u}\wedge\vec{B} = \frac{\mu_0 I}{4\pi}\; e\,\vec{u}\wedge\vec{e}_\theta$$

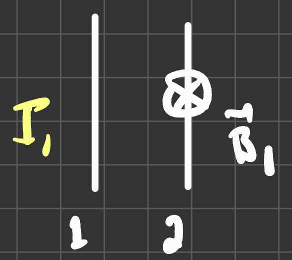
(If the particle is moving on a circle of radius R around the wire (ie $\vec{u} = u\,\hat{e}_\theta$) then the particle experiences no force)

Example : Calculate the force between two straight infinite parallel wires carrying steady currents I_1 & I_2



- each wire generates a magnetic field
- field generated by one exerts a force on the other

Force per unit length on wire-2 due to I_1 in wire-1



$$\vec{F}_{12} = \vec{I}_2 \wedge \vec{B}_1 \leftarrow \text{magnetic field due to } I_1 \text{ in wire-1}$$

$$= I_2 \hat{k} \wedge \frac{\mu_0 I_1}{2\pi R} \hat{e}_\theta = -\frac{\mu_0}{2\pi R} I_1 I_2 \hat{e}_R$$

Force per unit length on wire-1 due to I_2 in wire-2

$$\vec{F}_{21} = -\vec{F}_{12} = \vec{I}_1 \wedge \vec{B}_2$$

3.3 Differential equations for $\vec{B}$

Recall eqs for electrostatics:

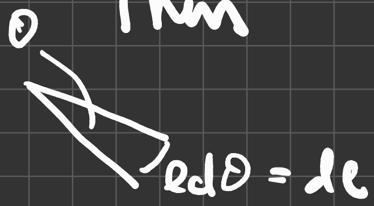
$$\nabla \cdot \vec{E} = \frac{1}{\epsilon_0} \rho \qquad \longleftrightarrow \qquad \nabla^2 \Phi = -\frac{1}{\epsilon_0} \rho, \quad \vec{E} = -\nabla \Phi$$

$$\nabla \wedge \vec{E} = 0$$

We want now to find equations satisfied by the magnetic field $\vec{B}$ given an arbitrary steady current density $\vec{J}$

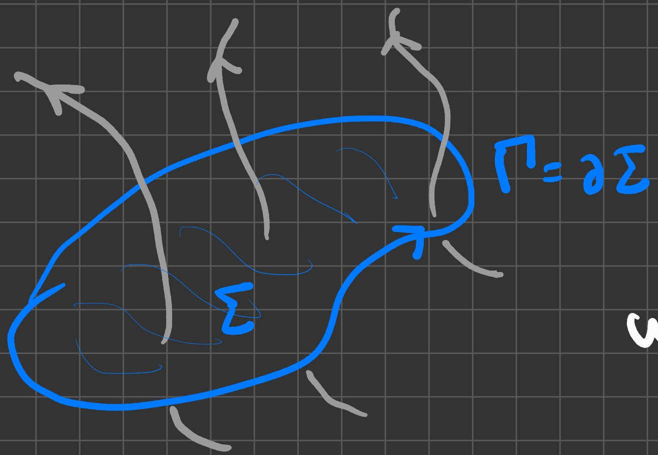
Consider an infinite straight wire along the z-axis carrying a constant current $\vec{I} = I\hat{z}$

Let Γ be a circle of radius R lying on the (x,y) plane and centered on the wire



Then $$\oint_\Gamma \vec{B}\cdot \underbrace{d\vec{\ell}}_{d\ell\,\hat{e}_\theta} = \frac{\mu_0 I}{2\pi}\int_0^{2\pi} \frac{1}{R}\,\underbrace{R\,d\theta}_{d\ell} = \mu_0 I$$

More generally



$$\boxed{\oint_\Gamma \vec{B}\cdot d\vec{\ell} = \mu_0 I}$$

Ampère law

where $I = \int_\Sigma \vec{J}\cdot d\vec{S}$ = total current through Σ

By Stoke's theorem

$$\oint_\Gamma \vec{B}\cdot d\vec{\ell} = \int_\Sigma (\nabla\wedge\vec{B})\cdot d\vec{S}$$

must be true $\forall\ \Sigma$

$$\parallel$$

$$\mu_0 I = \mu_0 \int_\Sigma \vec{J}\cdot d\vec{S}$$

$$\Rightarrow \quad \boxed{\nabla\wedge\vec{B} = \mu_0\vec{J}}$$

Ampère's law
($\vec{J}$ generates $\vec{B}$)

Remark: take the divergence of this equation gives

$$\nabla\cdot\vec{J} = 0$$

so the conservation eq is required for consistency.

No magnetic monopoles

$$\boxed{\nabla \cdot \vec{B} = 0}$$

- For the magnetic field generated by a steady current in an the infinite straight wire:

$$\nabla \cdot \vec{B} = \frac{1}{R} \frac{dB}{d\theta} = 0 \checkmark$$

- For the magnetic field generated by a charged particle moving with velocity $\vec{v}$: $\nabla \cdot \vec{B} = 0$, $\vec{r} \neq \vec{0}$

However: $$\int_S \vec{B} \cdot d\vec{S} = \frac{\mu_0 q}{4\pi} \int_S \frac{(\vec{v} \wedge \vec{r}) \cdot \hat{n} \, dS}{r^3} = 0$$

$(\vec{v} \wedge \vec{r})$ perpendicular to $\vec{r}$

S = sphere, center at origin, radius r

$$\Rightarrow \int_V \nabla \cdot \vec{B} \, dV = 0$$ for any region $V \subseteq \mathbb{R}^3$

[compare with coulomb's law for a point charge q at the origin

$$\nabla \cdot \vec{E} = 0 \qquad \vec{r} \neq \vec{0}$$

However

$$\int_S \vec{E} \cdot d\vec{S} = \frac{1}{\epsilon_0} q \qquad \forall\ \underline{0} \in V, \quad S = \partial V \quad]$$

▲ no magnetic monopoles have been found in nature !

there are magnetic dipoles !

Question : we say that there are no magnetic monopoles and that currents generate magnetic fields. Then, what does produce the magnetic field in permanent magnets?

- the alignment of the spin of the electrons in an atom (needs quantum mechanics):

Basically electrons in an atom have a magnetic dipole moment.

If a large number of the electrons have their spin aligned, a magnetic field is produced

Next

$\boxed{13.4}$ Differential equations for $\vec{B}$ in terms of a vector potential $\vec{A}$

↳ Maxwell's eqs for magnetostatics. in terms of $\vec{A}$

$\nabla \cdot \vec{B} = 0$

$\nabla \wedge \vec{B} = \mu_0 \vec{J}$

$\boxed{13.5}$ Multipole expansion

↳ what is the $\vec{E}$ or $\vec{B}$ of a localized ρ or $\vec{J}$ at large distances from the distribution?