B7.2 ELECTROMAGNETISM

Chapter 3: Magnetostatics (Part 2)

Lecture 10

3 Magnetostatics

$\boxed{13.4}$ The Magnetic potential $\vec{A}$ and the eqs of magnetostatics in terms of $\vec{A}$

Recall eqs for electrostatics:

$$\nabla \cdot \vec{E} = \frac{1}{\epsilon_0}\rho \qquad \longleftrightarrow \qquad \nabla^2 \Phi = -\frac{1}{\epsilon_0}\rho , \quad \vec{E} = -\nabla \Phi$$

$$\nabla \wedge \vec{E} = 0$$

Φ defined only up to a constant

For magnetostatics we have

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \wedge \vec{B} = \mu_0 \vec{J}$$

??

Consider $\nabla \cdot \vec{B} = 0$.

Then there exists $\vec{A}$ st $\boxed{\vec{B} = \nabla \wedge \vec{A}}$

$\vec{A}$: magnetic vector potential

$\vec{A}$ is not unique:

$$\vec{A} \longrightarrow \vec{A}' = \vec{A} + \nabla \chi$$

"gauge transformation"

Leaves $\vec{B}$ unchanged for any χ.

[$\vec{B} = \nabla \wedge \vec{A}' = \nabla \wedge \vec{A} \iff \nabla \wedge (\vec{A}' - A) = 0$ so $\vec{A}' - A = \nabla \chi$ for some χ]

We will exploit this freedom to simplify resulting equation for $\vec{A}$.

Now consider $\nabla \wedge \vec{B} = \mu_0 \vec{J}$ together with $\vec{B} = \nabla \wedge \vec{A}$ to find an equation for $\vec{A}$

$$\nabla \wedge (\nabla \wedge \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

Then

$$-\nabla^2 \vec{A} + \nabla(\nabla \cdot \vec{A}) = \mu_0 \vec{J} \qquad (*)$$

let $\vec{A}' = A + \nabla \chi$. Then

$$\nabla \cdot \vec{A}' = \nabla \cdot \vec{A} + \nabla^2 \chi$$

and $(*)$ is

$$\nabla \wedge \vec{B} = -\nabla^2 \vec{A}' + \nabla(\nabla \cdot \vec{A}') = \mu_0 \vec{J}$$

Choose χ such that $\nabla \cdot \vec{A}' = 0$

one can always do this

Then

$$-\nabla^2 \vec{A}' = \mu_0 \vec{J}$$

$$\nabla \cdot \vec{A}' = 0$$

Drop the primes :

$$\boxed{\begin{array}{c} -\nabla^2 \vec{A} = \mu_0 \vec{J} \\ \nabla \cdot \vec{A} = 0 \end{array}}$$

Poisson's eq for each component of $\vec{A}$

Lorentz gauge

(Lorenz not Lorentz)

Recall : a solution of $\nabla^2 \Phi = \frac{1}{\epsilon_0}\rho$ which $\Phi \to O(\frac{1}{r})$ as $r \to \infty$

is $$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{r}\,')\frac{1}{|\vec{r}-\vec{r}\,'|}\,dV'$$

It is not hard to see that for a current density $\vec{J}(\vec{r})$

$$\vec{B} = \frac{\mu_0}{4\pi}\int_V \vec{J}(\vec{r}\,') \wedge \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\,dV' = \nabla \wedge \vec{A}$$

where $$\boxed{\vec{A} = \frac{\mu_0}{4\pi}\int_V \vec{J}(\vec{r}\,')\frac{1}{|\vec{r}-\vec{r}\,'|}\,dV'}$$

but how about $\nabla \cdot \vec{A} = 0$?

One can prove that $\nabla\cdot\vec{A}=0$ if there are no currents at infinity:

$$\nabla\cdot\vec{A} = \frac{\mu_0}{4\pi}\int_V \nabla\cdot\left(\vec{J}(\vec{r}\,')\frac{1}{|\vec{r}-\vec{r}\,'|}\,dV'\right)$$

$$= -\frac{\mu_0}{4\pi}\int_V \vec{J}(\vec{r}\,')\cdot\nabla'\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)dV'$$

$$= -\frac{\mu_0}{4\pi}\int_V \left[\nabla'\cdot\left(\vec{J}(\vec{r}\,')\frac{1}{|\vec{r}-\vec{r}\,'|}\right) - \frac{1}{|\vec{r}-\vec{r}\,'|}\cancelto{0}{\nabla'\cdot\vec{J}(\vec{r}\,')}\right]dV'$$

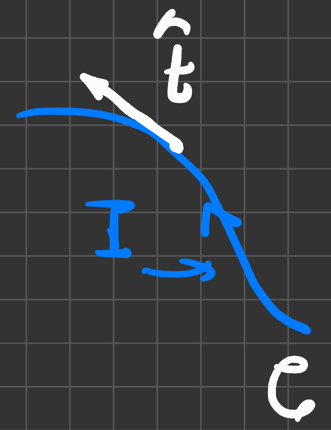
continuity equation

$$= -\frac{\mu_0}{4\pi}\oint_{S=\partial V}\frac{1}{|\vec{r}-\vec{r}\,'|}\vec{J}(\vec{r}\,')\cdot d\vec{S}\,'$$

$= 0$ if there are no currents at infinity or in fact outside V including $S=\partial V$

if the current is localized

Example: Consider a wire C with a line current distribution



$$\vec{J} = I\,\hat{t}$$

constant current along C

unit vector tangent to C at each point

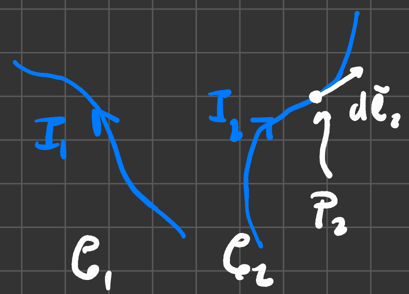
Suppose the curve C is parametrised by ℓ

Then $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_C \frac{I\, d\vec{\ell}'}{|\vec{r}-\vec{r}'|} = \frac{\mu_0 I}{4\pi}\int_C \frac{d\vec{\ell}'}{|\vec{r}-\vec{r}'|}$

$$\vec{B}(\vec{r}) = \nabla\wedge\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int_C \nabla\wedge\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) d\vec{\ell}' = \frac{\mu_0 I}{4\pi}\int_C d\vec{\ell}'\wedge\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|}$$

generally hard to evaluate!

Example: compute the force between two arbitrary wires $\mathcal{C}_1$ & $\mathcal{C}_2$, each carrying a current



$$\vec J_1 = I_1 \hat t_1 \qquad \vec J_2 = I_2 \hat t_2$$

with I_1 & I_2 constant.

Consider an element of wire in $\mathcal{C}_2$ at P_2 with tangent $d\vec\ell_2$. Then by the Lorentz force law

$$d\vec F = I_2 d\vec\ell_2 \wedge \vec B,$$

is the force on the element at P_2 due to the magnetic field generated by I_1 in $\mathcal{C}_1$. Integrating

$$\vec F = \int_{\mathcal{C}_2} I_2 d\vec\ell_2 \wedge \vec B_1 = \frac{\mu_0}{4\pi} I_1 I_2 \int_{\mathcal{C}_2}\int_{\mathcal{C}_1} d\vec\ell_2 \wedge d\vec\ell_1 \wedge \frac{\vec r_2 - \vec r_1}{|\vec r_2 - \vec r_1|}$$

total force on $\mathcal{C}_2$ due to $\vec B_1$ generated by I_1 on $\mathcal{C}_1$

Very hard to evaluate in general!

3.5 Multipole expansion

In this subsection we are interested in studying the electrostatic & magnetostatic fields of a localized charge distribution $\rho(\vec{r})$ or current distribution $\vec{J}(\vec{r})$

[What we are learning in this section applies in other areas, as for example the study of gravitational waves in GR]

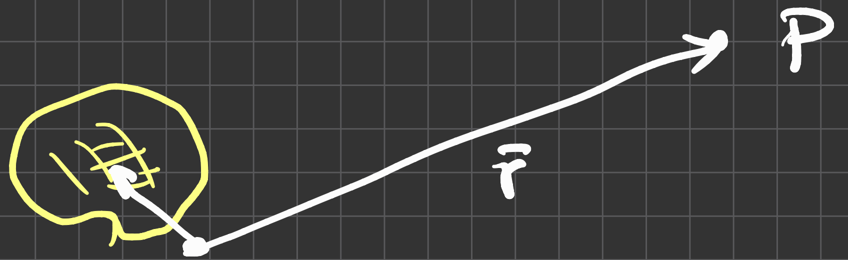
Consider the magnetic field $\vec{B}(\vec{r})$ due to a general current distribution $\vec{J}(\vec{r})$ localized in a small region of space V

small relative to the distance to the observation point

We know that the magnetostatic potential is

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V \frac{1}{|\vec{r}-\vec{r}\,'|} \vec{J}(\vec{r}\,')\, dV'$$

we want to evaluate the integral for $|\vec{r}| \gg |\vec{r}\,'|$



How? use Taylor series expansion of $\frac{1}{|\vec{r}-\vec{r}'|}$

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r} + \frac{1}{r^3}\,\vec{r}\cdot\vec{r}' + \Theta(r^{-3})$$

exercise $r \gg r'$ (verify)

This leads to the multipole expansion of $\vec{A}(\vec{r})$

sum of moments of $\vec{J}(\vec{r})$

We have then

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_V \left(\frac{1}{r} + \frac{1}{r^3}\,\vec{r}\cdot\vec{r}' + \Theta(r^{-3})\right)\vec{J}(\vec{r}')\,dV'$$

$$= \frac{\mu_0}{4\pi}\left\{\underbrace{\frac{1}{r}\int_V \vec{J}(\vec{r}')\,dV'}_{\text{no monopoles}} + \underbrace{\frac{1}{r^3}\int \vec{r}\cdot\vec{r}'\,\vec{J}(\vec{r}')\,dV'}_{\text{dipole}} + \underbrace{\cdots}_{\text{higher moments of } \vec{J}}\right\}$$

(yes: this vanishes! no monopoles ...)

Tool to compute this: (Jackson)

$$\int_V [f(\vec{r}')\,\vec{J}(\vec{r}')\cdot\nabla' g(\vec{r}') + g(\vec{r}')\,\vec{J}(\vec{r}')\cdot\nabla' f(\vec{r}')]\,dV' = 0$$

where f & g are arbitrary functions and $\vec{J}(\vec{r})$ is a localised vector field and satisfies

$$\nabla\cdot\vec{J}(\vec{r}) = 0 \quad !$$

▸ For $f = 1$ and $g = x_i$

$$\int_V \vec{J}(\vec{r}')\cdot\nabla' x_i' = \int_V J_i(\vec{r}')\,dV = 0$$

$$\Longrightarrow \int_V \vec{J}(\vec{r}')\,dV' = 0 \qquad \text{as expected.}$$

▶ For $f = x_i$ & $g = x_j$

$$\int x_i' \vec{J}(\vec{r}') \cdot \nabla' x_j' + x_j' \vec{J}(\vec{r}') \cdot \nabla' x_j$$

$$= \int \left(x_i' J_j(\vec{r}') + x_j' \vec{J}_i(\vec{r}') \right) = 0$$

so

$$\int_V \vec{r} \cdot \vec{r}' \vec{J}(\vec{r}')\, dv' = \sum_{i,j} x_i \left(\int_V x_i' J_j(\vec{r}')\, dv' \right) \hat{e}_i$$

$$= \frac{1}{2} \sum_{i,j} x_i \int_V \left(x_i' J_j(\vec{r}') - x_j' J_i(\vec{r}') \right) dv' \, \hat{e}_i$$

$$= -\frac{1}{2} \vec{r} \wedge \int_V \vec{r}' \wedge \vec{J}(\vec{r}')\, dv'$$

$$\vec{a} \wedge (\vec{b} \wedge \vec{c}) = -(\vec{a} \cdot \vec{b}) \vec{c} + (\vec{a} \cdot \vec{c}) \vec{b}$$

using this in $\vec{A}(\vec{r})$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\left\{ \frac{1}{r}\int_V \vec{J}(\vec{r}\,')\,dV' + \frac{1}{r^3}\int \vec{r}\cdot\vec{r}\,'\,\vec{J}(\vec{r}\,')\,dV' + \cdots \right\}$$

$$= \underbrace{0}_{\text{no monopoles}} + \frac{\mu_0}{4\pi}\left(-\frac{1}{2}\right)\vec{r}\wedge \underbrace{\int_V \vec{r}\,'\wedge\vec{J}(\vec{r}\,')\,dV'}_{\text{dipole moment of }\vec{J}} + \underbrace{\cdots}_{\text{higher moments of }\vec{J}}$$

which we can write as

$$\vec{A}(\vec{r}) = 0 + \underbrace{\frac{\mu_0}{4\pi}\frac{1}{r^3}\,\vec{m}\wedge\vec{r}}_{\text{magnetic dipole vector potential}} + \cdots\cdots$$

where

$$\vec{m} = \frac{1}{2}\int_V \underbrace{\vec{r}\,'\wedge\vec{J}(\vec{r}\,')\,dV'}_{\substack{\text{magnetization}\\ \text{(magnetic moment density)}}}$$

magnetic dipole moment of $\vec{J}$

We can now compute the magnetic field outside a localised source

$$\vec{B} = \nabla\wedge\vec{A} = \frac{\mu_0}{4\pi}\nabla\wedge\left(\vec{m}\wedge\frac{1}{r^3}\vec{r}\right)+\cdots$$

$$= 0 + \frac{\mu_0}{4\pi}\left\{\underbrace{\left(\nabla\cdot\left(\frac{1}{r^3}\vec{r}\right)\right)}_{0}\vec{m} - \vec{m}\cdot\nabla\left(\frac{1}{r^3}\vec{r}\right)\right\}+\cdots$$

$$= 0 + \underbrace{\frac{\mu_0}{4\pi}\frac{1}{r^3}\left(-\vec{m}+\frac{3}{r^2}(\vec{m}\cdot\vec{r})\vec{r}\right)}_{}+\cdots$$

far far away $\vec{B}=0$

field lines of $\vec{B}_{dipole}$ are the same as those for $\vec{E}_{dipole}$ when $\vec{m}\leftrightarrow\vec{p}$ and $\mu_0\leftrightarrow\frac{1}{\epsilon_0}$!

Field lines of $\vec{B}_{dipole}$ (same as for $\vec{E}_{dipole}$

with $\vec{m} \leftrightarrow \vec{p}$, $\mu_0 \leftrightarrow \frac{1}{\varepsilon_0}$)

Compare with the electric field $\vec{E}$ due to a localized charge distribution ρ

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{1}{|\vec{r}-\vec{r}'|} \rho(\vec{r}')\, dV'$$

$$= \frac{1}{4\pi\epsilon_0} \frac{q}{r} + \frac{1}{r^3} \vec{p}\cdot\vec{r} + \frac{1}{2r^5} \sum_{i,j} Q_{ij} x_i x_j + \cdots$$

$q = \int_V \rho(r')\, dV'$

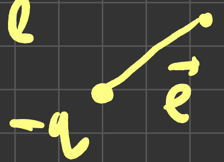
far far away the distribution "looks like" a point charge q

electric dipole

$\vec{p} = \int_V \vec{r}' \rho(\vec{r}')\, dV'$

gives $\vec{E}$ due to a dipole with dipole moment $\vec{p}$

(eg $\vec{p} = q\vec{\ell}$



quadrupole ...

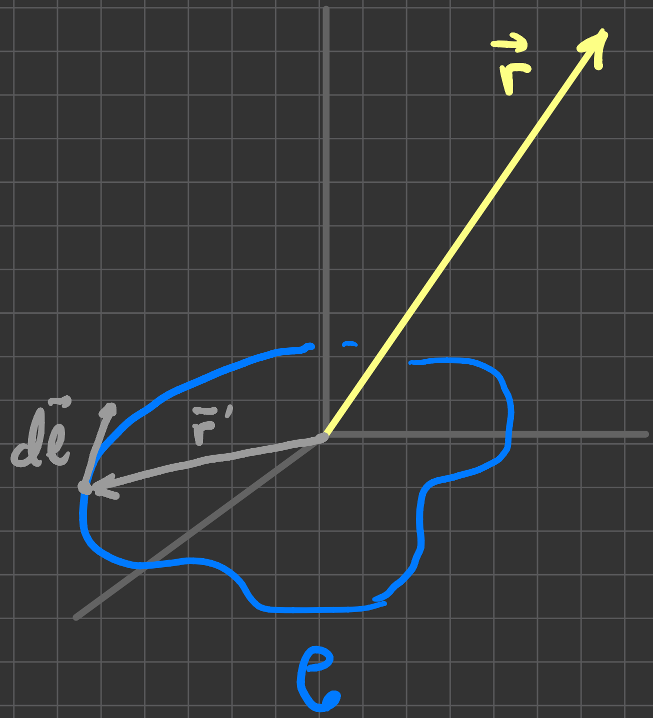
Example: Suppose the source of current is a distant circuit $\mathcal{C}$ carrying a constant current I

The potential is

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int_{\mathcal{C}} \frac{1}{|r-\vec{r}|} \, d\vec{\ell}'$$

hard

$$= \underbrace{\frac{\mu_0}{4\pi r^3} \, \vec{m} \wedge \vec{r}}_{\text{far away it "looks" like a magnetic dipole}} + \underbrace{\cdots}_{\text{needs more details about the geometry of } \mathcal{C}}$$



can we calculate $\vec{m}$?

$$\vec{m} = \frac{1}{2} \underset{\mathcal{C}}{\int_V} \vec{r}\,' \wedge \underbrace{\vec{J}(\vec{r}\,')\, dV'}_{I\, d\vec{\ell}} = \frac{I}{2} \oint_{\mathcal{C}} \vec{r}\,' \wedge d\vec{\ell}$$

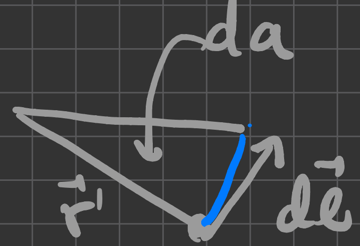
Assume the wire lies on a plane.

Then $\vec{m}$ is perpendicular to the plane of the wire

Now note That

$$\frac{1}{2} |\vec{r} \wedge d\vec{\ell}| = da$$

triangular element of area



$$\Longrightarrow \quad |\vec{m}| = I \cdot \text{Area}$$

regardless of the shape of the circuit

Higher moments of $\vec{J}$ need more detailed knowledge of the geometry of $\mathcal{C}$

3.C Epilogue

- The magnetic field of a magnet exerts a force on certain material as for example "ferromagnetic" materials

  eg iron

A magnet has a magnetic field mainly due to its magnetic dipole moment

▷ Compare the magnitudes of magnetic fields of different objects:

- refrigerator magnet ~10^{-3} Tesla
- superconducting magnets in CERN (dipole magnets) ~8 Tesla
- Earth's magnetic field (~ magnetic dipole) 25-65 10^{-6} Tesla on the surface
    - generated by Earth's core of liquid iron
    - shields against radiation from the sun by deflecting cosmic rays and charged particles in solar wind (stream of charged particles released from the sun)
    - auroras

- MRI ~1.5 - 3 Teslas

- Sun ~ $2\times B_{earth}$

  not solid; plasma, perfect conductor

How do we know? For example:

Zeeman effect to measure magnetic fields

splitting of light into "components" in the presence of a static field

- Neutron stars $10^4 - 10^{11}$ Tesla !

↳ some of the most fascinating objects in the universe

very small $m \sim 1.5\, M_{sun}$ $r \sim 10\, km$

smallest densest stars, almost entirely composed of neutrons (neutral particles in the nucleons of atoms with $m_{neutron} \sim m_{proton}$)

new neutron stars rotate ~100 times per sec (linear surface speed c/4) $c = 300\,000\ km/sec$

(compare: the sun rotation period ~ 25 days at its equator)

<u>Next</u>: time varying ρ & $\vec{J}$ and
Maxwell's equations.