B7.2 ELECTROMAGNETISM

Chapter 4 : Maxwells equations (part 1)

Lecture 11

$\boxed{4}$ Maxwell's equations

(and time varying fields)

$\boxed{4.1}$ Maxwell's equations

We already have two (scalar) equations which are valid when the fields and charge and current densities vary with time

$$\nabla \cdot \vec{E} = \frac{1}{\epsilon_0} \rho \qquad \text{Gauss' law}$$

$$\nabla \cdot \vec{B} = 0 \qquad \text{no magnetic monopoles}$$

The two other equations discussed in the static setting

$$\nabla \wedge \vec{E} = 0 \quad , \quad \nabla \wedge \vec{B} = \mu_0 \vec{J}$$

need to be amended. In fact one finds experimentally that

a varying $\vec{E}$ produces $\vec{B}$

a varying $\vec{B}$ produces $\vec{E}$

Ampère's law and Maxwell's displacement current

Recall: a steady current density $\vec{J}(\vec{r})$ produces a magnetic field which satisfies

$$\nabla \wedge \vec{B} = \mu_0 \vec{J} \qquad \text{(Ampère)}$$

This equation requires for consistency

$$\nabla \cdot \vec{J} = 0 \qquad \left(\text{continuity equation when } \frac{\partial \rho}{\partial t} = 0\right)$$

Maxwell added a term to Ampère's law

$$\nabla \wedge \vec{B} + \vec{X} = \mu_0 \vec{J}$$

with $\vec{X}$ such that the new equation be consistent with the continuity eq $\quad \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0$

Taking the divergence of $\quad \nabla \wedge \vec{B} + \vec{X} = \mu_0 \vec{J}$

$$\nabla \cdot (\cancel{\nabla \wedge \vec{B}} + \vec{X}) = \mu_0 \nabla \cdot \vec{J}$$

$$\nabla \cdot \vec{X} = -\mu_0 \frac{\partial \rho}{\partial t}$$

Now take $\partial/\partial t$ of Gauss' law

$$\nabla \cdot \frac{\partial \vec{E}}{\partial t} = \frac{1}{\epsilon_0} \frac{\partial \rho}{\partial t}$$

Then

$$\nabla \cdot \vec{X} = -\mu_0 \epsilon_0 \nabla \cdot \frac{\partial \vec{E}}{\partial t}$$

ie

$$\nabla \cdot \left(\vec{X} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} \right) = 0 \qquad \text{where } \boxed{c^2 = \frac{1}{\mu_0 \epsilon_0}}$$

So $\vec{X} = -\frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} + \nabla \wedge \vec{\alpha}$ for some $\vec{\alpha}$

↳ redefine what you mean by $\vec{B}$

and $\nabla \wedge \vec{B} + \vec{X} = \mu_0 \vec{J}$ becomes

$$\boxed{\nabla \wedge \vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}}$$

Ampère-Maxwell

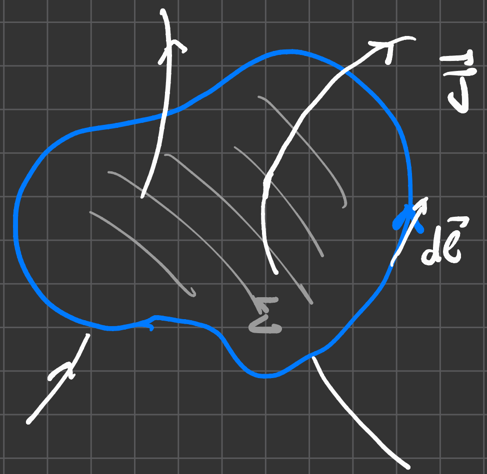
exp. indeed the correct time dependent eq

↳ Maxwell displacement current

Time dependent $\vec{E}$ produces $\vec{B}$ even in the absence of currents.

Integral form:

Consider the flux of a current $\vec{J}(\vec{r})$ through a surface Σ with $\mathcal{C} = \partial\Sigma$ a simple closed loop bounding Σ



$$\int_{\Sigma} (\nabla \wedge \vec{B}) \cdot d\vec{S} = \oint_{\mathcal{C}} \vec{B} \cdot d\vec{\ell}$$

Stoke's

Then Ampère-Maxwell equation becomes

$$\oint_{\mathcal{C}} \vec{B} \cdot d\vec{\ell} = \int_{\Sigma} \left(\mu_0 \vec{J} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} \right) \cdot d\vec{S}$$

ie

$$\oint_{\mathcal{C}} \vec{B} \cdot d\vec{\ell} = \mu_0 I + \frac{1}{c^2} \int_{\Sigma} \frac{\partial \vec{E}}{\partial t} \cdot d\vec{S}$$

rate of change of flux of $\vec{E}$ through Σ

Note that even if $\vec{J} = 0$, a wire in the presence of a varying $\vec{E}$ induces a magnetic field along the wire.

Faraday's law of magnetic induction

Recall: $\nabla \wedge \vec{E} = 0$ electrostatics

This equation is replaced by

$$\boxed{\nabla \wedge \vec{E} + \frac{\partial}{\partial t} \vec{B} = 0}$$

Faraday

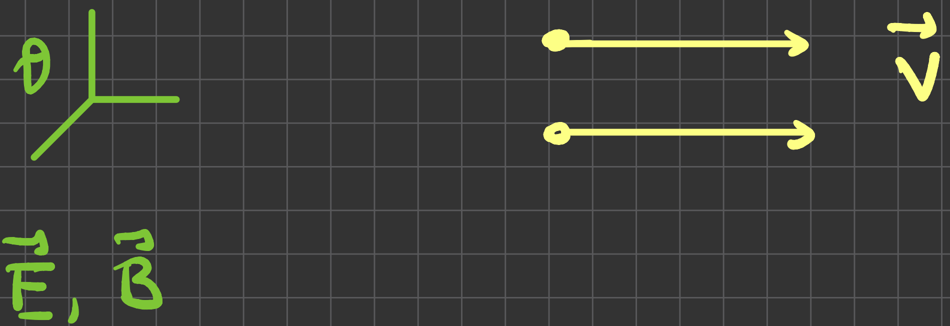
so a varying magnetic field produces an electric field.

This term needs to be there so that Maxwell's equations are consistent with special relativity. (Maxwell didn't know this !!)

We will return to this in Chapter 6 when we discuss electromagnetism & special relativity.

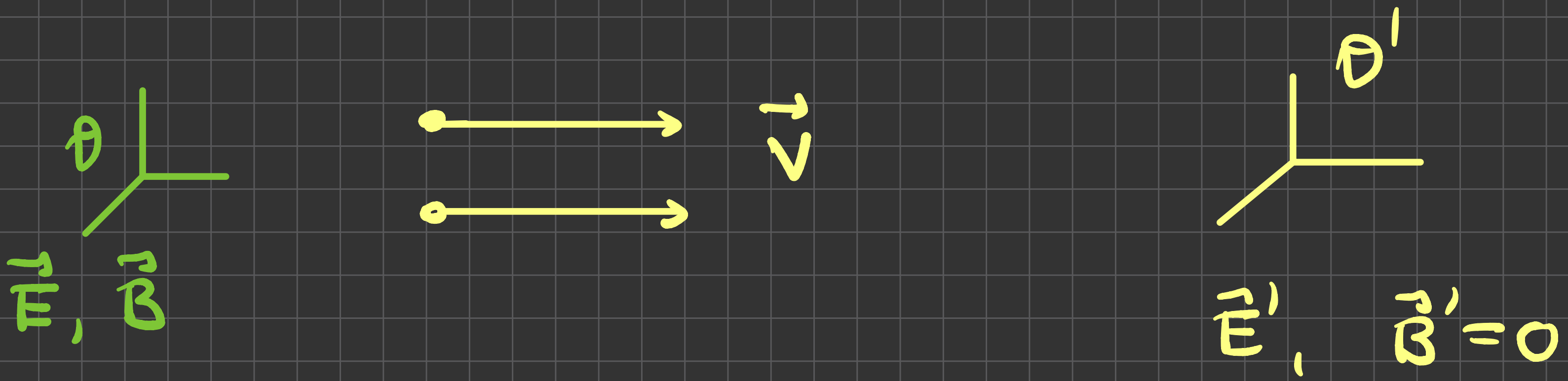
For the time being notice what happens, in a simple example, when we make measurements with respect to two reference frames moving relative to each other at speeds $v \ll c$ so that we can use Galilean relativity.

Consider a set of moving charges all moving with the same constant velocity $\vec{v}$ with respect to an observer $\mathcal{O}$



$\mathcal{O}$ says these charges generate both, an electric field $\vec{E}$ and a magnetic field $\vec{B}$

Consider now an observer $\mathcal{O}'$ which moves at the same constant velocity $\vec{v}$ as the charges:



this observer will says that $\vec{B}'=0$ and she will <u>only</u> measure an electric field $\vec{E}'$

Assume they both measure the same force on a test particle

$$\vec{E}' = \vec{E} + \vec{v} \wedge \vec{B}$$

For the moving observer $\mathcal{O}'$

$\nabla' \wedge \vec{E}' = \nabla \wedge \vec{E}' = 0 \Rightarrow \quad 0 = \nabla \wedge \vec{E} + \nabla \wedge (\vec{v} \wedge \vec{B})$

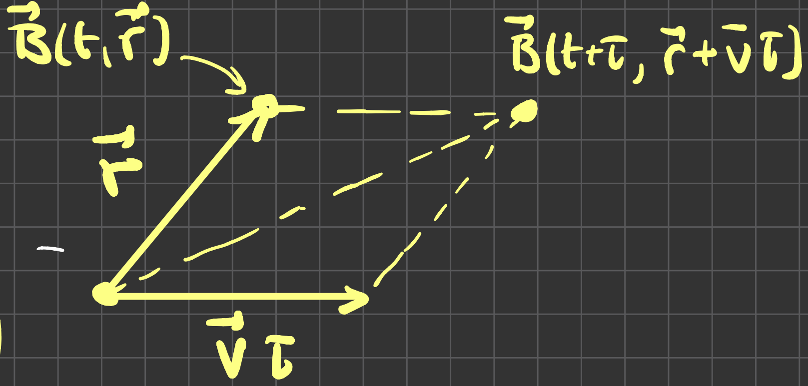
(valid in electrostatics)

$$= \nabla \wedge \vec{E} + \vec{v}\,(\cancel{\nabla \cdot \vec{B}}) - (\vec{v} \cdot \nabla)\,\vec{B}$$

For the observer $\mathcal{O}$

$$\vec{B}(t+\tau, \vec{r} + \vec{v}\tau) = \vec{B}(t, \vec{r})$$

($\vec{B}$ only depends on the distance between the particle and the point of observation)



Then

$$0 = \frac{d\vec{B}}{dt} = \frac{\partial \vec{B}}{\partial t} + \sum_{i=1}^{3} \frac{\partial x^i}{\partial t} \frac{\partial \vec{B}}{\partial x^i} = \frac{\partial \vec{B}}{\partial t} + (\vec{v}\cdot\nabla)\vec{B}$$

so $$0 = \nabla \wedge \vec{E} - (\vec{v}\cdot\nabla)\vec{B} = \nabla \wedge \vec{E} + \frac{\partial \vec{B}}{\partial t} \checkmark$$

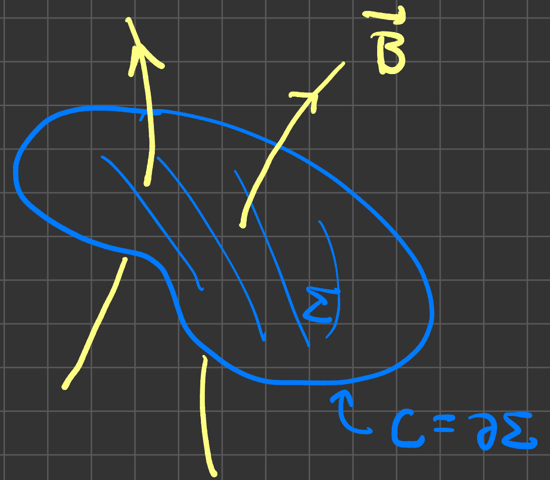
We will do this properly in Chapter 6

Integral form: consider the flux of $\vec{B}$ through a surface Σ bounded by $\mathcal{C} = \partial\Sigma$

$$\int_\Sigma (\nabla \wedge \vec{E}) \cdot d\vec{S} = -\frac{d}{dt}\int_\Sigma \vec{B} \cdot d\vec{S}$$

Stoke's $\quad \parallel$

$$\oint_C \vec{E} \cdot d\vec{\ell} = -\frac{d}{dt}\int_\Sigma \vec{B} \cdot d\vec{S}$$



work done by $\vec{E}$ (per unit charge) around the loop C

rate of change of the magnetic flux through Σ

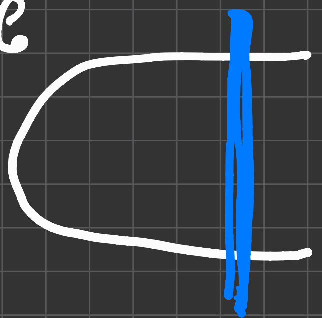
↳ conversion of electromagnetic energy into mechanic energy

If $\mathcal{C}$ is a conducting wire a current would be induced on $\mathcal{C}$!

To get a better understanding of the meaning of Faraday's law we consider the following example.

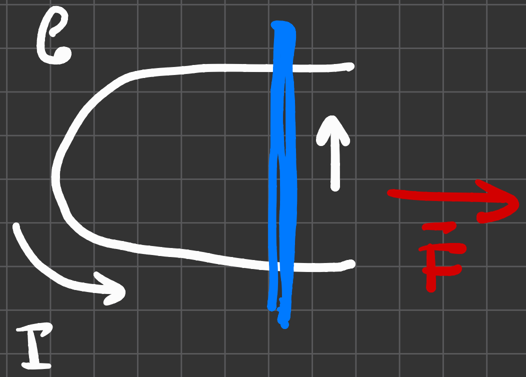
Suppose we have a wire $\mathcal{C}$ lying on the (x,y) plane which makes an open loop. Place a conducting bar on the wire so that it closes the loop as in the picture.

$\mathcal{C}$



Now turn on a varying magnetic field $\vec{B}$.

What happens to the bar?

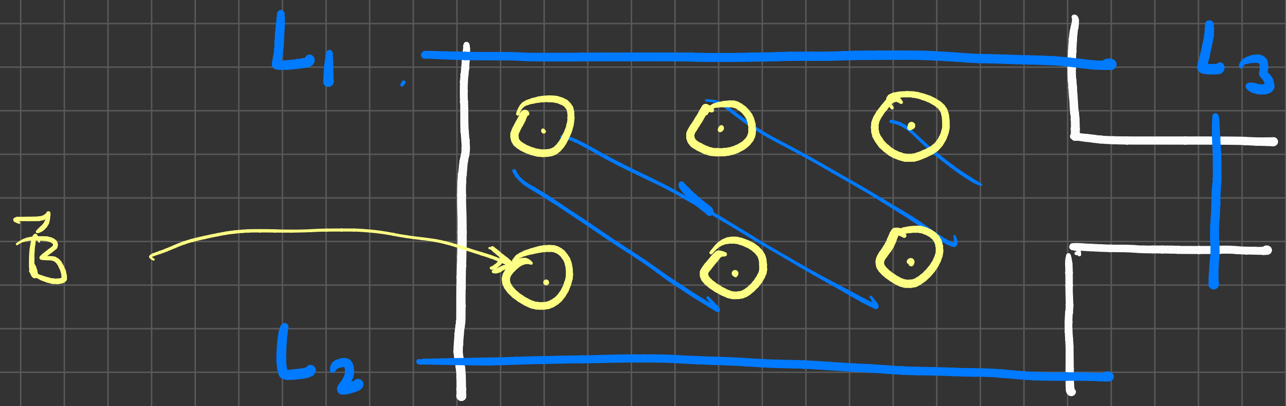


Faraday's law implies that a current is created around the closed loop, which means charged particles move along the wire with some velocity $\vec{v}$.

By Lorentz force law, a magnetic force acts on these moving charges.

Then the bar experiences a force $\vec{F}$ perpendicular to the bar.

Alternatively, consider the following circuit $\mathcal{C}$

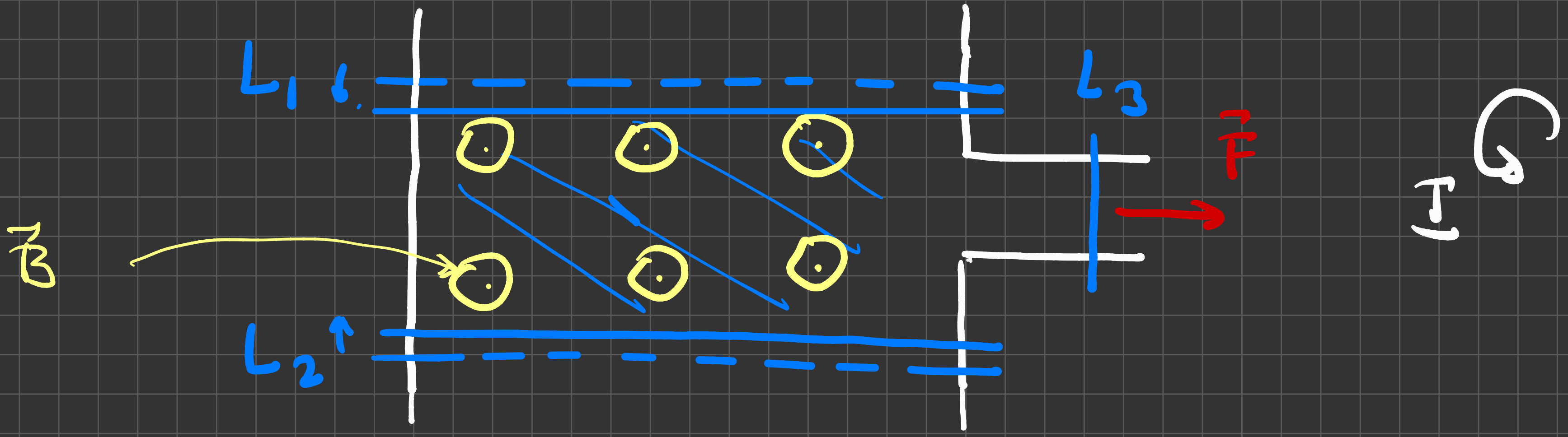


$$\partial \Sigma = \mathcal{C}$$

where the bars in blue (L_1, L_2, L_3) are free to move.
Let Σ be the surface enclosed by $\mathcal{C}$.

Turn on a static magnetic field $\vec{B}$ perpendicular (say) to the plane formed by the circuit.

Now start moving the bars L_1 & L_2.
What happens to the bar L_3?



In this case though $\vec{B}$ is not varying but the flux of $\vec{B}$ through the now varying surface enclosed by the wires, therefore a current is created along the wire.

As a consequence the bar L_3 starts moving due to the magnetic force on the charges.
(Recommended: Feynman lectures Vol 2)

Maxwell's equations

$\nabla \cdot \vec{E} = \frac{1}{\epsilon_0} \rho$	Gauss
$\nabla \wedge \vec{E} + \frac{\partial}{\partial t} \vec{B} = 0$	Faraday
$\nabla \cdot \vec{B} = 0$	no magnetic monopole
$\nabla \wedge \vec{B} - \frac{1}{c^2} \frac{\partial}{\partial t} \vec{E} = \mu_0 \vec{J}$	Ampère-Maxwell

where $c^2 = 1/\mu_0 \epsilon_0$. Together with Lorentz Force law

$$\vec{F} = q(\vec{E} + \vec{v} \wedge \vec{B})$$

they describe electromagnetic phenomena.

Some remarks:

① 2 scalar equations + 2 vector equations

$\Rightarrow$ 8 equations for 6 unknowns $\vec{E}$ & $\vec{B}$ (given $\rho, \vec{J}$)

which seems to be an overdetermined system

Consistency: the conservation of charge

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0$$

is implicit in Maxwell's equations. In fact, it is required for consistency.

[Proof: $\dfrac{\partial \rho}{\partial t} \overset{\text{Gauss}}{=} \epsilon_0 \nabla \cdot \dfrac{\partial \tilde{E}}{\partial t} \overset{\text{Ampère}}{=} \epsilon_0 c^2 \nabla \cdot \left(\cancel{\nabla \wedge \vec{B}} - \mu_0 \vec{J} \right) = -\nabla \cdot \vec{J}$ ↘

Also $0 = \nabla \cdot \dfrac{\partial \vec{B}}{\partial t} \underset{\text{Faraday}}{=} -\nabla \cdot (\nabla \wedge \vec{E}) = 0$ ↘]

But we knew this! the displacement current was added precisely to achive consistency with the conservation of charge equation.

(2) Maxwell's equations are a set of PDEs for $\vec{E}$ & $\vec{B}$ given $\rho, \vec{J}$

However, when there are no sources ($\rho=0, \vec{J}=0$) they have non-trivial solutions: electromagnetic waves (light)

Next:

electromagnetic potentials, $(\Phi, \vec{A})$

energy of the electromagnetic fields