B7.2 ELECTROMAGNETISM

Chapter 4 : Maxwells equations (part 2)

Lecture 12

$\boxed{4}$ Maxwell's equations

Last lecture

$\boxed{4.1}$ Maxwell's equations

$\nabla \cdot \vec{E} = \frac{1}{\epsilon_0} \rho$	Gauss
$\nabla \cdot \vec{B} = 0$	no magnetic monopoles
$\nabla \wedge \vec{E} + \frac{\partial}{\partial t} \vec{B} = 0$	Faraday's law of induction
$\nabla \wedge \vec{B} - \frac{1}{c^2} \frac{\partial}{\partial t} \vec{E} = \mu_0 \vec{J}$	Ampère-Maxwell

$$c^2 = 1/\mu_0 \epsilon_0$$

This lecture

$\boxed{4.2}$ Electromagnetic potentials $(\Phi, \vec{A})$

↳ Maxwell's eqs in terms of $(\Phi, \vec{A})$

$\boxed{4.3}$ Energy of the electromagnetic field and Poynting's theorem

↳ conservation of energy

4.2 Electromagnetic potentials

We now write Maxwell's equations for $\vec{E}$ & $\vec{B}$ in terms of electromagnetic potentials

Let V be a suitable region in space

← simply connected, ...

From $\nabla \cdot \vec{B} = 0$ we write $\boxed{\vec{B} = \nabla \wedge \vec{A}}$ for some vector field $\vec{A}$

From Faraday's law

$$0 = \nabla \wedge \vec{E} + \frac{\partial \vec{B}}{\partial t} = \nabla \wedge \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0$$

$\exists \quad \Phi \quad st$

$$\vec{E} + \frac{\partial}{\partial t}\vec{A} = -\nabla \Phi$$

Then there is a function Φ st: $\boxed{\vec{E} = -\nabla \Phi - \frac{\partial \vec{A}}{\partial t}}$

The other two eqs give equations for Φ & $\vec{A}$.

$\vec{A}$ & Φ are not uniquely defined

Let $(\Phi_1, \vec{A}_1)$ & $(\Phi_2, \vec{A}_2)$ be electromagnetic potentials which leave $\vec{E}$ & $\vec{B}$ invariant

$$\vec{B}_1 = \vec{B}_2 \qquad \& \qquad \vec{E}_1 = \vec{E}_2$$

Then ▸ $\vec{B}_1 = \vec{B}_2 \Leftrightarrow \nabla \wedge (\vec{A}_2 - \vec{A}_1) = 0$

so $\vec{A}_2 = \vec{A}_1 + \nabla \chi$ for some function χ

▸ $\vec{E}_1 = \vec{E}_2 \Leftrightarrow -\nabla \Phi_1 - \dfrac{\partial \vec{A}_1}{\partial t} = -\nabla \Phi_2 - \dfrac{\partial \vec{A}_2}{\partial t}$

$$\Leftrightarrow \nabla \left(\underbrace{\Phi_2 - \Phi_1 + \frac{\partial \chi}{\partial t}}_{f(t)} \right) = 0$$

so $\Phi_2 = \Phi_1 - \dfrac{\partial \chi}{\partial t} + f(t)$

absorb into χ without affecting $\vec{A}_2 = \vec{A}_1 + \nabla \chi$

We define a gauge transformation of $(\Phi, \vec{A})$ as a change of $(\Phi, \vec{A})$ which leaves the electromagnetic fields $\vec{E}$ & $\vec{B}$ invariant

$$\boxed{\begin{aligned} \Phi &\longrightarrow \Phi' = \Phi - \frac{\partial \chi}{\partial t} \\ \vec{A} &\longrightarrow \vec{A}' = \vec{A} + \nabla \chi \end{aligned}}$$

We use this freedom to define $(\Phi, \vec{A})$ to simplify the equations for Φ & $\vec{A}$.

Equations for $(\Phi, \vec{A})$:

From Gauss law:

$$\frac{1}{\epsilon_0}\rho = \nabla\cdot\vec{E} = \nabla\cdot\left(-\nabla\Phi - \frac{\partial}{\partial t}\vec{A}\right)$$

so $$\nabla^2\Phi + \frac{\partial}{\partial t}\nabla\cdot\vec{A} = -\frac{1}{\epsilon_0}\rho$$

From Ampère-Maxwell:

$$\mu_0\vec{J} = \nabla\wedge\vec{B} - \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} = \underbrace{\nabla\wedge(\nabla\wedge\vec{A})}_{-\nabla^2\vec{A}+\nabla(\nabla\cdot\vec{A})} - \frac{1}{c^2}\frac{\partial}{\partial t}\left(-\nabla\Phi - \frac{\partial}{\partial t}\vec{A}\right)$$

$$= -\nabla^2\vec{A} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} + \nabla\left(\nabla\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t}\right)$$

We have then coupled differential equations for $(\Phi, \vec{A})$

$$-\frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} + \nabla^2 \Phi + \frac{\partial}{\partial t}\left(\nabla\cdot\vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t}\right) = -\frac{1}{\varepsilon_0}\rho$$

$$-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} + \nabla^2\vec{A} - \nabla\left(\nabla\cdot\vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t}\right) = -\mu_0\vec{J}$$

Choosing $(\Phi, \vec{A})$ st

$$\boxed{\nabla\cdot\vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0}$$

Lorentz gauge

$$\Rightarrow \boxed{\begin{aligned} \Box\Phi &= -\frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} + \nabla^2\Phi = -\frac{1}{\varepsilon_0}\rho \\ \Box\vec{A} &= -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} + \nabla^2\vec{A} = -\mu_0\vec{J} \end{aligned}}$$

inhomogeneous wave eqs !

waves travelling with speed c

$$\Box = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2$$

d'Alambertian or wave operator

4 eqs for 4 unknowns

Of course a solution of these equations give $\vec{E}$ & $\vec{B}$ from

$$\vec{B} = \nabla \wedge \vec{A}$$

$$\vec{E} = -\nabla \Phi - \frac{\partial \vec{A}}{\partial t}$$

<u>Remark</u>: the Lorentz gauge is consistent with the charge conservation equation

$$0 = \frac{\partial}{\partial t}\rho + \nabla \cdot \vec{J} = -\epsilon_0 \frac{\partial}{\partial t} \Box \Phi - \frac{1}{\mu_0} \nabla \cdot \Box \vec{A}$$

$$= -\frac{1}{\mu_0} \Box \left(\nabla \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} \right) = 0 \quad \checkmark$$

14.3 Energy of the electromagnetic field and Poynting's theorem

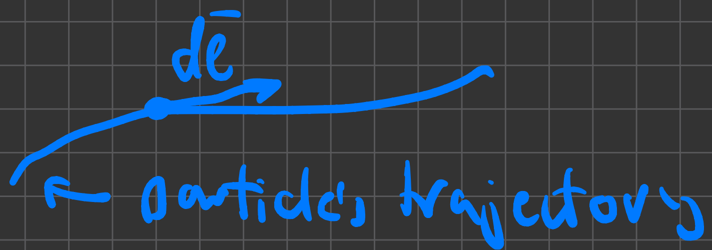
Consider a single charge q moving with velocity $\vec{v}$ in an electromagnetic field with $\vec{E}$ & $\vec{B}$.
The charge experiences a force

$$\vec{F} = q(\vec{E} + \vec{v} \wedge \vec{B})$$

The work done by the electromagnetic force in moving a particle a distance $d\vec{\ell}$ is

$$dW = \vec{F} \cdot d\vec{\ell}$$

$d\vec{\ell}$ = displacement tangent to the particle's trajectory



The rate of doing work by the external electromagnetic fields is then

$$\frac{dW}{dt} = \vec{F}\cdot\vec{v} = q(\vec{E} + \cancel{\vec{v}\wedge\vec{B}})\cdot\vec{v} = q\,\vec{E}\cdot\vec{v}$$

(the magnetic field does no work)

For a volume distribution of charges & currents in a region V we have

$$\frac{dW}{dt} = \int_V \vec{J}\cdot\vec{E}\,dV$$

total rate of doing work by the fields in a finite region V

represents the conversion of electromagnetic energy into mechanical energy (Kinetic energy)

As the electromagnetic field does work on the charge & current distribution, then the electromagnetic energy decreases.

But we expect energy to be conserved

so the rate of doing work must be balanced by the corresponding decrease of energy in the electromagnetic fields in V

We have

$$\int_V \vec{J}\cdot\vec{E}\,dV = \int \frac{1}{\mu_0}\left(\nabla\wedge\vec{B} - \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}\right)\cdot\vec{E}\,dV$$

use Ampère's law to eliminate $\vec{J}$

$$-\frac{1}{2c^2}\frac{\partial}{\partial t}|\vec{E}|^2$$

$$(\nabla\wedge\vec{B})\cdot\vec{E} = (\nabla\wedge\vec{E})\cdot\vec{B} - \nabla\cdot(\vec{E}\wedge\vec{B})$$

Faraday's

$$= \left(-\frac{\partial\vec{B}}{\partial t}\right)\cdot\vec{B} - \nabla\cdot(\vec{E}\wedge\vec{B})$$

$$= -\frac{1}{2}\frac{\partial}{\partial t}|\vec{B}|^2 - \nabla\cdot(\vec{E}\wedge\vec{B})$$

$$\int_V \vec{J}\cdot\vec{E}\,dV = \int\left(-\frac{1}{\mu_0}\nabla\cdot(\vec{E}\wedge\vec{B}) - \frac{1}{2\mu_0}\frac{\partial}{\partial t}|\vec{B}|^2 - \frac{1}{2c^2}\frac{\partial}{\partial t}|\vec{E}|^2\right)dV$$

Thus

$$\frac{dW}{dt} = \int_V \vec{J}\cdot\vec{E}\, dV = -\int_V \left(\nabla\cdot\vec{P} + \frac{\partial}{\partial t}\mathcal{E} \right) dV$$

$$= -\int_{S=\partial V} \vec{P}\cdot d\vec{S} + \underbrace{\frac{d}{dt}\int_V \mathcal{E}\, dV}_{\text{rate of change of energy stored in } V}$$

rate of energy density flowing out through $S=\partial V$ per unit time

(electromagnetic energy stored in V)

where

$$\mathcal{E} = \frac{1}{2}\left(\epsilon_0 |\vec{E}|^2 + \frac{1}{\mu_0}|\vec{B}|^2 \right)$$

total electromagnetic energy density in V

$$\vec{P} = \frac{1}{\mu_0}\vec{E}\wedge\vec{B}$$

Poynting's vector

momentum density of electromagnetic fields

As this must be true for any arbitrary region V

$$\boxed{\vec{J}\cdot\vec{E} = -\underbrace{\left(\nabla\cdot\vec{P} + \frac{\partial \mathcal{E}}{\partial t}\right)}_{\text{rate of decrease of electromagnetic energy in } V}}$$

Poynting's theorem

work done by the electromagnetic field

[Note $\vec{J} = 0$: continuity equation for electromagnetic energy]

Question: why do you get hot when standing under the sun?

electromagnetic waves coming from the sun!

electromagnetic waves which are not reflected are absorbed.

Absorbed electromagnetic waves transfer their energy to your skin increasing temperature (see chapter 5!)

<u>Next</u>: solving the inhomogeneous wave eq using Green's functions and Fourier integrals.

14.4 Time dependent Green's functions

(want skills to solve inhomogeneous wave equations + important examples -- radiation)

let ψ st

$$\Box\psi = -\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} + \nabla^2\psi = -4\pi f(t,\vec{r})$$

We want to find solutions using Green's functions.

<u>Definition</u>: a Green's function $G(t,\vec{r},t',\vec{r}')$ satisfies

$$\boxed{\Box G(t,\vec{r},t',\vec{r}') = -4\pi\,\delta(t-t')\,\delta(\vec{r}-\vec{r}')}$$

Remark: comparing with electrostatics

Recall that in electrostatics we considered Green's functions satisfying

$$\nabla^2 G^2(\vec{r}, \vec{r}') = -4\pi\,\delta(\vec{r}-\vec{r}')$$

$G(\vec{r}, \vec{r}')$ can be interpreted as the electrostatic potential of a source at $\vec{r}-\vec{r}'$. For example

$\dfrac{1}{\vec{r}-\vec{r}'}$ is the electrostatic potential for a source of charge $q = 4\pi\epsilon_0$ at $\vec{r}-\vec{r}'$

For time-varying fields we have

$$\Box G = 0 \quad \text{when} \quad \vec{r} = \vec{r}' \ \& \ t = t'$$

wave equation

However:

$$\int_{t \in \mathbb{R}} dt' \int_{\vec{r} \in V} dV' \, \Box G = -4\pi$$

Then we can interpret $G(t, \vec{r}, t', \vec{r}')$ as the electromagnetic potential of a wave caused by a source (of some electromagnetic disturbance) at the point $\vec{r} = \vec{r}'$ when $t = t'$

Consider a configuration where there are no boundaries

so G depends only on $\vec{r}-\vec{r}'$ & $t-t'$ (spherical symmetry)

We will find G in terms of orthonormal functions

Recall the integral representation for the Dirac δ-function, that is, the Fourier integral of the δ-function

$$\delta(t-t')\,\delta(\vec{r}-\vec{r}\,') = \underbrace{\frac{1}{(2\pi)^4}}_{\text{(completeness)}} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d^3k \; \underbrace{e^{-i\omega(t-t')}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')}}_{\text{complete set of orthonormal exponentials}}$$

The Fourier integral for $G(t-t', \vec{r}-\vec{r}\,')$ is

$$G(t-t', \vec{r}-\vec{r}\,') = \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d^3k \; \underbrace{g(\omega, \vec{k})}_{\text{Fourier transform of } G} \, e^{-i\omega(t-t')}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')}$$

Using the Fourier integrals for the δ-function and the Green's function G into Green's equation we hope to find the Fourier transform g of G

$$\Box G(t-t', \vec{r}-\vec{r}')$$

$$= \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d^3k \, g(\omega, \vec{k}) \, \underbrace{\Box\left(e^{-i\omega(t-t')} e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}\right)}_{= \left(\frac{\omega^2}{c^2} - |\vec{k}|^2\right) e^{-i\omega(t-t')} e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}}$$

$$\|$$

$$-4\pi \, \delta(t-t') \, \delta(\vec{r}-\vec{r}') = -\frac{4\pi}{(2\pi)^4} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d^3k \, e^{-i\omega(t-t')} e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}$$

We can now read off the Fourier transform

$$g(\omega, \vec{k}) = \frac{1}{4\pi^3} \frac{1}{k^2 - \omega^2/c^2}$$

$k = |\vec{k}|$

Therefore the Green's function is

$$G(t-t', \vec{r}-\vec{r}') = \frac{1}{4\pi^3} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d^3k \, \frac{1}{k^2 - \omega^2/c^2} \, e^{-i\omega(t-t')} \, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}$$

two poles: $\omega = \pm kc$

Jackson:

- circuit with a constant current I
- suppose flux of $\vec{B}$ through circuit changes

$$W = \oint_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int_\Sigma \vec{B} \cdot d\vec{S}$$

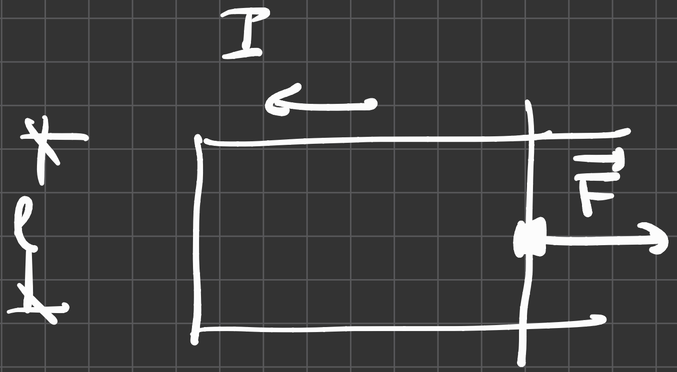
what is the induced current on I?

Recall $$I = \int_\Sigma \vec{J} \cdot d\vec{S}$$

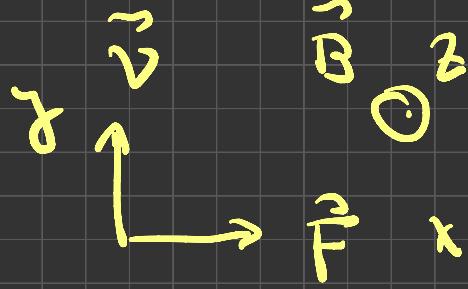
$$\mu_0 I = \int_\Sigma \left(\nabla \wedge \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} \right) \cdot d\vec{S}$$

$$= \int_C \vec{B} \cdot d\vec{l} - \frac{1}{c^2}$$

$$\mathcal{E} = -\frac{d\Phi}{dt} = \oint \vec{E}\cdot d\vec{\ell} = -\frac{d}{dt}\int_{\Sigma} \vec{B}\cdot d\vec{S}$$



$\odot\ \vec{B}$



$$d\vec{F} = \delta q\ \vec{v} \wedge \vec{B}$$

$$F = I\ell B$$