B7.2 ELECTROMAGNETISM

Chapter 4 : Maxwells equations (part 3)

Lecture 13

$\boxed{4}$ Maxwell's equations (and time varying fields)

$\boxed{4.1}$ Maxwell's equations

$\boxed{4.2}$ Electromagnetic potentials $(\Phi, \vec{A})$

$\boxed{4.3}$ Energy of the electromagnetic field and Poynting's theorem

$\boxed{4.4}$ Time dependent Green's functions

4.4 Time dependent Green's functions

Last lecture: we can find the electromagnetic fields $\vec{E}$ and $\vec{B}$ in terms of the electromagnetic potentials $(\Phi, \vec{A})$

$$\vec{E} = -\nabla \Phi - \frac{\partial \vec{A}}{\partial t} \qquad \vec{B} = \nabla \wedge \vec{A}$$

where in the Lorenz gauge $\left(\frac{1}{c^2}\frac{\partial \Phi}{\partial t} + \nabla \cdot \vec{A} = 0\right)$ the potentials satisfy

$$\left.\begin{aligned} \Box \Phi &= -\frac{1}{\epsilon_0} \rho \\ \Box \vec{A} &= -\mu_0 \vec{J} \end{aligned}\right\}$$

inhomogeneous wave eqs!

where $\Box = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2$ (d'Alembertian or wave operator)

(want skills to solve inhomogeneous wave equations + important examples -- radiation)

Let $\Psi(t, \vec{r})$ satisfy $\Box \Psi = -4\pi f(t, \vec{r})$

We want to find solutions for $\Psi(t, \vec{r})$ using Green's functions (as in electrostatics).

Definition: a Green's function $G(t, \vec{r}, t', \vec{r}')$ satisfies

$$\boxed{\Box G(t, \vec{r}, t', \vec{r}') = -4\pi \delta(t-t') \delta(\vec{r}-\vec{r}')}$$

We assume that G depends only on $\vec{r}-\vec{r}'$ & $t-t'$, Considering a configuration where there are no boundaries, a solution for ψ in terms of the Green's function is then

$$\psi(t,\vec{r}) = \int_{V} G(t,\vec{r},t',\vec{r}')\, f(t',\vec{r}')\, dV'dt'$$

$$\left[\ \Box\psi = \int_{V} (-4\pi\, \delta(t-t')\delta(\vec{r}-\vec{r}'))\, f(t',\vec{r}')\, dV' = -4\pi f(t,\vec{r})\ \checkmark\ \right]$$

where the Green's function needs to satisfy appropriate conditions depending on physical constraints on ψ

(for example "initial" conditions when $t \to -\infty$ ie in the "past", etc)

Remark: comparing with electrostatics

Recall that in electrostatics we considered Green's functions satisfying

$$\nabla^2 G^2(\vec{r}, \vec{r}') = -4\pi\, \delta(\vec{r} - \vec{r}')$$

$G(\vec{r}, \vec{r}')$ can be interpreted as the electrostatic potential of a source at $\vec{r}'$. For example

$\dfrac{1}{\vec{r} - \vec{r}'}$ is the electrostatic potential for a source of charge $q = 4\pi\varepsilon_0$ at $\vec{r}'$

For time-varying fields we have

$\Box G = 0$ when $\vec{r} \neq \vec{r}'$ & $t \neq t'$

↳ wave equation → waves travelling at speed c

However: $$\int_{t \in \mathbb{R}} dt' \int_{\vec{r} \in V} dV' \, \Box G = -4\pi$$

Thus we can interpret $G(t, \vec{r}, t', \vec{r}')$ as the electromagnetic potential of a wave caused by a source (or some electromagnetic disturbance) at the point $\vec{r} = \vec{r}'$ when $t = t'$ travelling as a spherical wave at velocity c

We find G in terms of orthonormal functions

Recall the integral representation for the Dirac δ-function, that is, the Fourier integral of the δ-function

$$\delta(t-t')\,\delta(\vec{r}-\vec{r}\,') \underset{\text{(completeness)}}{=} \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d^3k\; \underbrace{e^{-i\omega(t-t')}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')}}_{\text{complete set of orthonormal exponentials}}$$

The Fourier integral for $G(t-t', \vec{r}-\vec{r}\,')$ is

$$G(t-t', \vec{r}-\vec{r}\,') = \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d^3k\; \underbrace{g(\omega, \vec{k})}_{\text{Fourier transform of } G}\, e^{-i\omega(t-t')}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')}$$

Using the Fourier integrals for the δ-function and the Green's function G into Green's equation we hope to find the Fourier transform g of G

$$\Box G(t-t', \vec{r}-\vec{r}') = \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d^3k \, g(\omega, \vec{k}) \underbrace{\Box\left(e^{-i\omega(t-t')} e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}\right)}_{= \left(\frac{\omega^2}{c^2} - |\vec{k}|^2\right) e^{-i\omega(t-t')} e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}}$$

$$\Box = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2$$

$$\|$$

$$-4\pi\,\delta(t-t')\,\delta(\vec{r}-\vec{r}') = -\frac{4\pi}{(2\pi)^4} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d^3k \, e^{-i\omega(t-t')} e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}$$

We can now read off the Fourier transform

$$\underline{g(\omega, \vec{k}) = \frac{1}{4\pi^3} \frac{1}{k^2 - \omega^2/c^2}} \qquad k = |\vec{k}|$$

Therefore the Green's function is

$$G(t-t', \vec{r}-\vec{r}') = \frac{1}{4\pi^3} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d^3k \, \frac{1}{k^2 - \omega^2/c^2} \, e^{-i\omega(t-t')} e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}$$

two poles: $\omega = \pm kc$!

Recall now that G represents a wave caused by a source at $\vec{r}'$ at $t'=t$ propagating with velocity c. We can then assume that

$$\underline{G(t-t', \vec{r}-\vec{r}') = 0 \quad \forall \; t < t'}$$

ie before the electromagnetic disturbance happens when $t = t'$

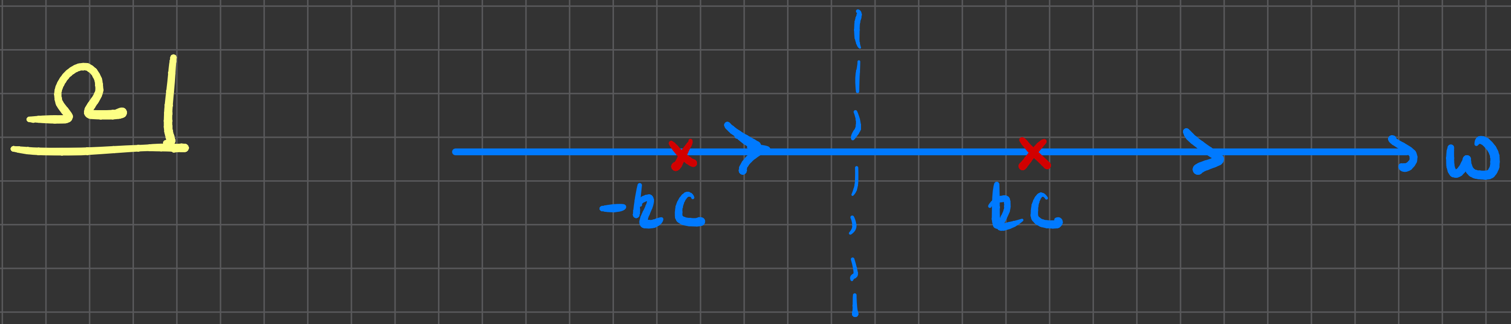
Then we use the expression above to compute G for $t > t'$.

compute first the integral over ω:

$$I(k, t-t') = \int_{-\infty}^{\infty} d\omega \, \frac{1}{k^2 - \omega^2/c^2} \, e^{-i\omega(t-t')}$$

We need to deal with the poles!

Consider the complex plane Ω with $\omega = \mathrm{Re}\,\Omega$



poles $\omega = \pm kc$

basic strategy:

$$\oint_C \cdots = \int_{-\infty}^{\infty} + \int_\Gamma \cdots$$

use the residue theorem

gives I

close the contour st this vanishes

Consider thus the integral

$$\tilde{I}(k, t-t') = \oint_{C} d\Omega \, \frac{1}{k^2 - \Omega^2/c^2} \, e^{-i\Omega(t-t')}$$

closed contour

$$= I(k, t-t') + \int_{\Gamma} d\Omega \, \frac{1}{k^2 - \Omega^2/c^2} \, e^{-i\Omega(t-t')}$$

choice of Γ dictated by

→ convergence

→ physical considerations

$\mathcal{C}$ must be such that it gives back $I(k, t-t')$

with $I(k, t-t') = 0 \quad t < t'$

Note that $e^{-i\Omega(t-t')} = e^{-i\omega(t-t')}\, e^{(\operatorname{Im}\Omega)(t-t')}$

For $\underline{t > t'}$: close at ∞ from below

with Γ a large semicircle

so $\int_\Gamma \to 0$ as $\operatorname{Im}\Omega \to -\infty$

For $\underline{t < t'}$: close at ∞ from above

with Γ a large semicircle

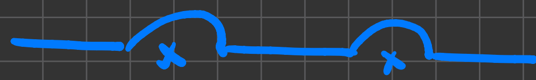
so $\int_\Gamma \to 0$ as $\operatorname{Im}\Omega \to \infty$

For $t < t'$ we require $I(k, t-t') = 0$

The contour C is

Γ = large semicircle with $|\Omega|$ fixed & then $|\Omega| \to \infty$

equivalently



$\downarrow$

poles $\Omega = \pm kc - i\epsilon$ and then $\epsilon \to 0$

Then

$$\tilde{I}(k, t-t') = \oint_C d\Omega \, \frac{1}{k^2 - \Omega^2/c^2} \, e^{-i\Omega(t-t')} = 0$$

by the residue theorem

$$= I(k, t-t') + \cancel{\int_\Gamma \cdots} \to 0$$

✓

For $t > t'$: The contour is

poles
$\Omega = \pm kc - i\epsilon$
and then $\epsilon \to 0$

Γ = large semicircle
with $|\Omega|$ fixed & then $|\Omega| \to \infty$

Then

$$\tilde{I}(k, t-t') = -I(k, t-t') + \int_\Gamma \cdots \to 0$$

$$= \oint_C d\Omega \, \frac{e^{-i\Omega(t-t')}}{k^2 - (\Omega + i\epsilon)^2/c^2}$$

$$= \frac{c^2}{2ck} \oint_C d\Omega \left\{ -\frac{1}{-ck + (\Omega + i\epsilon)} + \frac{1}{ck + (\Omega + i\epsilon)} \right\} e^{-i\Omega(t-t')}$$

pole at $\Omega = ck - i\epsilon$

pole at $\Omega = -ck - i\epsilon$

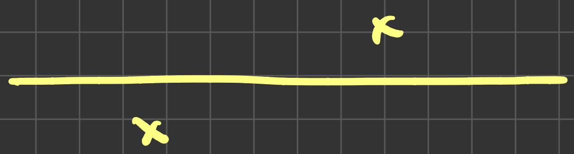
$$\tilde{I}(k, t-t') = \frac{c}{2k} 2\pi i \left\{ - e^{-ick(t-t')} + e^{ick(t-t')} \right\}$$

$$= \frac{c}{2k} 2\pi i \; 2i \sin(ck(t-t'))$$

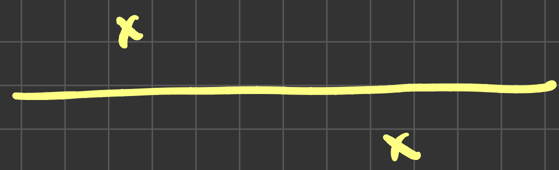
$$= -2\pi \frac{c}{k} \sin(ck(t-t'))$$

$$\Rightarrow \quad I(k, t-t') = 2\pi \frac{c}{k} \sin(ck(t-t')), \quad t > t'$$

Remark: you might want to check that



or



doesn't work!

Going back to the Green's function

$$G(t-t', \vec{r}-\vec{r}') = \frac{1}{4\pi^3} \int_{-\infty}^{\infty} d^3k \; I(k, t-t') \, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}$$

$$= \frac{1}{4\pi^3} \, 2\pi c \int_{-\infty}^{\infty} d^3k \; \underbrace{\frac{1}{k} \sin(ck(t-t'))}_{\text{depends on } k \text{ only}} \, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}$$

$k = |\vec{k}|$

using "spherical" coordinates for $\vec{k}$

$$d^3k = k^2 dk \sin\theta \, d\theta \, d\phi$$

$$G(t-t', \vec{r}-\vec{r}') = \frac{c}{2\pi^2} \int_0^{\infty} dk \; k \sin(ck(t-t'))$$

$$\times \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi \, e^{ik|\vec{r}-\vec{r}'|\cos\theta}$$

compute angular integrals first

$$G(t-t', \vec{r}-\vec{r}') = \frac{c}{2\pi^2} \int_0^\infty dk \, \cancel{k} \sin(ck(t-t'))$$

$$\times \; 2\pi \cdot \frac{2}{\cancel{k}} \frac{1}{|\vec{r}-\vec{r}'|} \sin(k|\vec{r}-\vec{r}'|)$$

$$= \frac{2c}{\pi} \frac{1}{|\vec{r}-\vec{r}'|} \int_0^\infty dk \underbrace{\sin(ck(t-t'))\sin(k|\vec{r}-\vec{r}'|)}_{\text{even function of } k}$$

$$= \frac{c}{\pi} \frac{1}{|\vec{r}-\vec{r}'|} \int_{-\infty}^\infty dk \underbrace{\sin(ck(t-t'))}_{\frac{1}{2i}\left(e^{ick(t-t')} - e^{-ick(t-t')}\right)} \underbrace{\sin(k|\vec{r}-\vec{r}'|)}_{\frac{1}{2i}\left(e^{ik|\vec{r}-\vec{r}'|} - e^{-ik|\vec{r}-\vec{r}'|}\right)}$$

$$G(t-t', \vec{r}-\vec{r}') = -\frac{1}{2\pi|\vec{r}-\vec{r}'|}\int_{-\infty}^{\infty} dk\, c \left\{ e^{ick\left(t-t'+\frac{1}{c}|\vec{r}-\vec{r}'|\right)} - e^{ick\left(t-t'-\frac{1}{c}|\vec{r}-\vec{r}'|\right)} \right\}$$

Recall: $\delta(x-x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x')}\,dk$, etc

$\Rightarrow$

$$G(t-t', \vec{r}-\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|}\left\{ \cancelto{0}{-\delta\left(t'-\frac{1}{c}|\vec{r}-\vec{r}'|-t\right)} + \delta\left(t'+\frac{1}{c}|\vec{r}-\vec{r}'|-t\right) \right\}$$

$\Big[\delta = 0$ unless

$t - t' = \underbrace{-\frac{1}{c}|\vec{r}-\vec{r}'|}_{\text{but this is negative}} \Big]$

Finally: $$\boxed{G(t-t', \vec{r}-\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|}\,\delta\left(t' + \frac{1}{c}|\vec{r}-\vec{r}'| - t\right)}$$

Note that $G = 0$ unless $|\vec{r}-\vec{r}'| = c(t-t')$

ie on the sphere of radius $c(t-t')$ at each t

G corresponds to a wave originating from an electromagnetic perturbation at $\vec{r} = \vec{r}'$ when $t = t'$ expanding spherically at velocity c.

In fact

$\frac{1}{c}|\vec{r}-\vec{r}'| = \Delta t = t - t' =$ time taken for a disturbance to reach the point $\vec{r}$ travelling at velocity c

G is the effect observed at $(t, \vec{r})$

caused by the source at a distance $|\vec{r}-\vec{r}'|$ away

earlier at a time

$$t' = t - \frac{1}{c}|\vec{r}-\vec{r}'| \qquad (t > t')$$

G $\rightarrow$ retarded Green's function

(causal behaviour corresponding to a wave disturbance)

in the past

Using Green's function we can now write down the solution to Poisson's equations

$$\longrightarrow \quad \Box \psi = -4\pi f(t, \vec{r})$$

ie $$\psi(t, \vec{r}) = \iint_V G(t-t', \vec{r}-\vec{r}')\, f(t', \vec{r}')\, dt'\, dV'$$

$$= \iint_V \frac{1}{|\vec{r}-\vec{r}'|}\, \delta\left(t' + \tfrac{1}{c}|\vec{r}-\vec{r}'| - t\right) f(t', \vec{r}')\, dt'\, dV'$$

integrate with respect to t'

$$= \int_V \frac{1}{|\vec{r}-\vec{r}'|}\, f\left(t - \tfrac{1}{c}|\vec{r}-\vec{r}'|, \vec{r}'\right) dV'$$

Hence the electromagnetic potentials in the absence of boundaries are given by

$$\Phi(t,\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{1}{|\vec{r}-\vec{r}'|}\,\rho\left(t-\frac{1}{c}|\vec{r}-\vec{r}'|\right) dV'$$

$$\vec{A}(t,\vec{r}) = \frac{\mu_0}{4\pi}\int_V \frac{1}{|\vec{r}-\vec{r}'|}\,\vec{J}\left(t-\frac{1}{c}|\vec{r}-\vec{r}'|, \vec{r}'\right) dV'$$

Exercise: check that this is consistent with the Lorenz gauge

$$\frac{1}{c^2}\frac{\partial \Phi}{\partial t} + \nabla\cdot\vec{A} = 0$$

Example: Liénard-Wiechert potentials

Consider a charged particle moving in a trajectory with charge q and motion $\vec{r}_0(t)$. Then

$$\rho(t,\vec{r}) = q\,\delta(\vec{r}-\vec{r}_0)$$

$$\vec{J}(t,\vec{r}) = q\,\vec{v}_0(t)\,\delta(\vec{r}-\vec{r}_0), \qquad \vec{v}_0(t) = \frac{d}{dt}\vec{r}_0(t)$$

Then

$$\Phi(t,\vec{r}) = \frac{q}{4\pi\epsilon_0}\iint_V \frac{1}{|\vec{r}-\vec{r}'|}\,\delta\left(t'+\tfrac{1}{c}|\vec{r}-\vec{r}'|-t\right)\delta(\vec{r}'-\vec{r}_0(t'))\,dt'\,dV'$$

$$= \frac{q}{4\pi\epsilon_0}\int dt'\,\frac{1}{|\vec{r}-\vec{r}_0(t')|}\,\delta\Big(\underbrace{t'+\tfrac{1}{c}|\vec{r}-\vec{r}_0|-t}_{\substack{\text{function of } t' \\ f(t';t,\vec{r}_0)}}\Big)$$

Using the property of the Dirac δ-function

$$\int dt \; g(t)\, \delta(f(t)) = \left\{ \frac{g(t)}{\left| \frac{df}{dt} \right|} \right\}_{f(t)=0}$$

evaluating at values of t which are the zeros of f

and setting $\vec{R}(t') = \vec{r} - \vec{r}_0(t')$

we have $f(t') = t' + \frac{1}{c}R - t$ $\qquad R = |\vec{R}|$

$$\frac{dR(t')}{dt'} = \frac{d}{dt'} |\vec{r} - \vec{r}_0(t')| = \frac{1}{2R} \frac{d}{dt'} R^2$$

$$= \frac{1}{2R} \frac{d}{dt'} (\vec{r} - \vec{r}_0(t')) \cdot (\vec{r} - \vec{r}_0(t'))$$

$$= -\frac{1}{R} (\vec{r} - \vec{r}_0(t')) \cdot \vec{V}_0(t') = -\frac{1}{R} \vec{R} \cdot \vec{V}_0(t')$$

$$\Phi(t, \vec{r}) = \frac{q}{4\pi\epsilon_0} \left\{ \frac{1}{R(t')} \times \frac{1}{1 - \frac{1}{c}\frac{\vec{R}(t')}{R(t')} \cdot \vec{v}_0(t')} \right\}_{\substack{t' \text{ st} \\ f(t')=0}}$$

similarly

$$\vec{J}(t, \vec{r}) = \frac{\mu_0 q}{4\pi} \left\{ \frac{\vec{v}_0(t')}{R(t')} \times \frac{1}{1 - \frac{1}{c}\frac{\vec{R}(t')}{R(t')} \cdot \vec{v}_0(t')} \right\}_{\substack{t' \text{ st} \\ f(t')=0}}$$

where $f(t') = t' + \frac{1}{c}R - t$

- We can now compute the electromagnetic field of the moving point charge

[accelerating charged particles produce electromagnetic radiation]

Next: Maxwell's eqs in macroscopic/dielectric media