B7.2 ELECTROMAGNETISM

Chapter 4 : Maxwells equations (part 4)

Lecture 14

$\boxed{4}$ Maxwell's equations (and time varying fields)

$\boxed{4.1}$ Maxwell's equations

$\boxed{4.2}$ Electromagnetic potentials $(\Phi, \vec{A})$

$\boxed{4.3}$ Energy of the electromagnetic field and Poynting's theorem

$\boxed{4.4}$ Time dependent Green's functions

$\boxed{4.5}$ Maxwell's equations in macroscopic and dielectric media

4.5 Maxwell's equations in macroscopic and dielectric media

Examples of macroscopic media are

- conductors : free charges (electrons)
- dielectric materials: charged particles are strongly bounded to their molecules

reality check: in practice

- how do we determine precisely what ρ & $\vec{J}$ are?
- how do we include the effects of applying electromagnetic fields to different substances?

Suppose we want to find solutions $\vec{E}$ of

$$\nabla \wedge \vec{E} = 0 \qquad \text{and} \qquad \epsilon_0 \nabla \cdot \vec{E} = \rho$$

for an electrostatic configuration of charges ρ.
So we need to know $\rho(\vec{r})$ everywhere in space.

In practice this is not possible except in very simple situations (eg discrete distribution of charges).

There is however an effective set of equations which describes electrostatics in macroscopic media

The bottom line is that one only needs to replace $\epsilon_0 \longrightarrow \epsilon$

in Maxwell's eqs in such a way that the medium itself does not contribute to ρ.

The claim is that Maxwell's equations are instead

$$\nabla \wedge \vec{E} = 0 \qquad \nabla \cdot (\epsilon \vec{E}) = \rho \qquad \text{(electrostatics)}$$

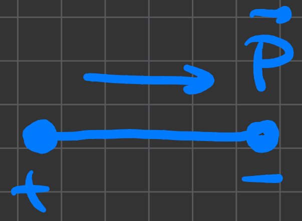
where $\frac{\epsilon}{\epsilon_0}$ is called the dielectric constant (or relative permittivity)

↳ measures the response to an applied $\vec{E}$

- vacuum $\epsilon = \epsilon_0$
- other substances $\frac{\epsilon}{\epsilon_0} > 1$ (air $\frac{\epsilon}{\epsilon_0} = 1.0005$

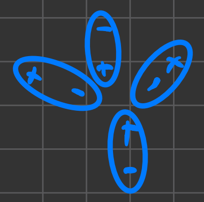
  water $\frac{\epsilon}{\epsilon_0} \sim 2$)
- in many cases $\frac{\epsilon}{\epsilon_0} =$ constant

Recall: an electric dipole with dipole moment $\vec{p}$ (consists of two opposite charges close to each other. The vector $\vec{p}$ is aligned in the direction of the line through their positions)



Macroscopic media contain dipoles which are randomly aligned (in equilibrium) when there are no external electric fields applied.

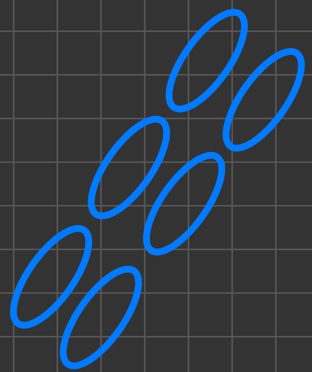
$\vec{E}_{applied} = 0$



in equilibrium

However when applying an electric field $\vec{E}_{applied}$ the molecules (dipoles) align

$\vec{E}_{applied} \neq 0$



molecules (dipoles) align

$\Rightarrow$ an electric field is produced in response to $\vec{E}_{applied}$

so we have instead $\vec{E}_{net} \propto \vec{E}_{applied}$

Thus, effectively $\epsilon_0 \rightarrow \epsilon$ and giving $\nabla \cdot (\epsilon \vec{E}) = \rho$

One can solve

$$\nabla \wedge \vec{E} = 0 \quad \text{and} \quad \nabla \cdot (\epsilon \vec{E}) = \rho$$

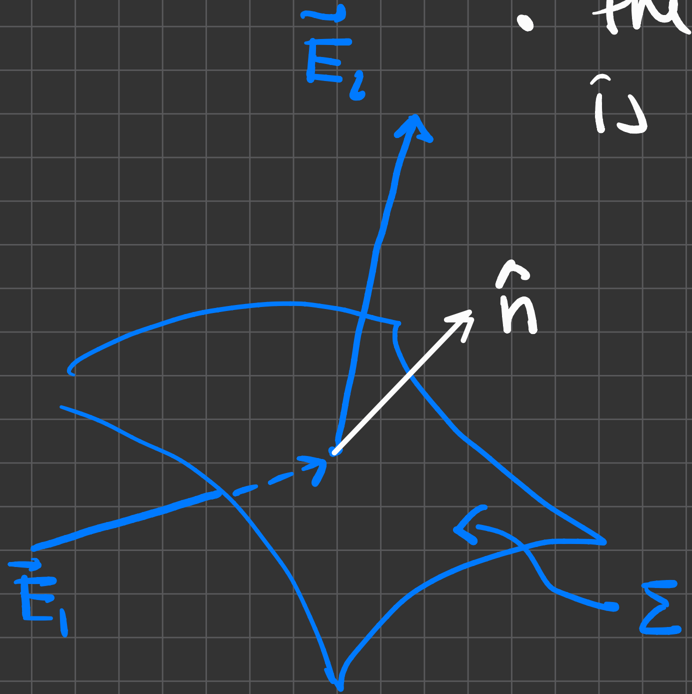
using the same tools as before.

BUT there is an important exception when there are boundaries between materials with different values of ϵ.

The claim we made in lecture 3 needs amendment

Claim: (lecture 3)

- the components of $\vec{E}$ tangent to Σ are continuous across Σ (proof from $\nabla\wedge\vec{E}=0$)
- the component of $\vec{E}$ normal to Σ is not continuous and



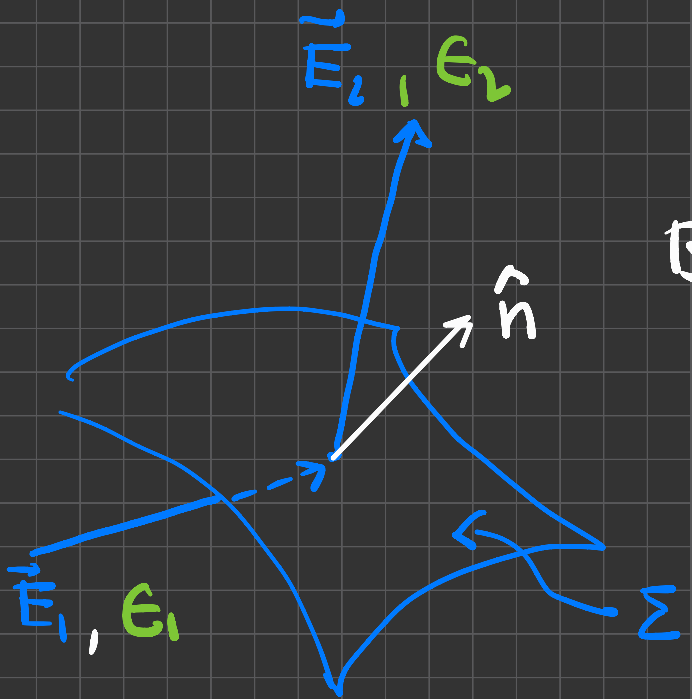
$$\boxed{\vec{E}_2 \cdot \hat{n} - \vec{E}_1 \cdot \hat{n} = \frac{\sigma}{\epsilon_0}}$$

proof: Gauss law

$\hat{n}$ = normal to Σ in the direction of $\vec{E}$ in region 2

Assumes that the substances on either side of the surface have $\epsilon = \epsilon_0$

Consider instead: Σ is the boundary between materials with different dielectric constant



By Gauss' law $\nabla \cdot (\epsilon \vec{E}) = 0$:

the component of $\epsilon\vec{E}$ normal to Σ is not continuous and

$$\boxed{\epsilon_2 \vec{E}_2 \cdot \hat{n} - \epsilon_1 \vec{E}_1 \cdot \hat{n} = \sigma}$$

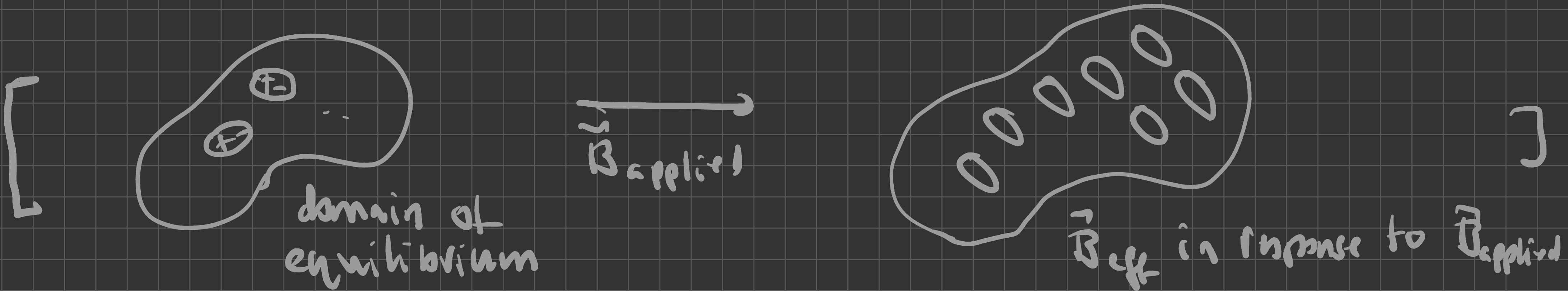
On the other hand (using $\nabla \wedge \vec{E} = 0$) we still find that the components of $\vec{E}$ tangent to Σ are continuous across Σ ($(\vec{E}_2 - \vec{E}_1) \wedge \hat{n} = 0$ on Σ)

For magnetostatic configurations the effective equations are

$$\nabla \cdot \vec{B} = 0 \qquad \nabla \wedge \left(\frac{1}{\mu} \vec{B} \right) = \vec{J}$$

$\dfrac{\mu}{\mu_0}$ permeability of the medium and measures the response to applied magnetic fields and the ability to get magnetized

Again μ is not a constant in general



- $\frac{\mu}{\mu_0} = 1$ vacuum
- $\frac{\mu}{\mu_0} \sim 1$ air (>1), water (<1)
- $\frac{\mu}{\mu_0} = 200\,000$ iron

  (ferromagnetic materials magnetize even under a small applied $\vec{B}$ and then stay magnetized)

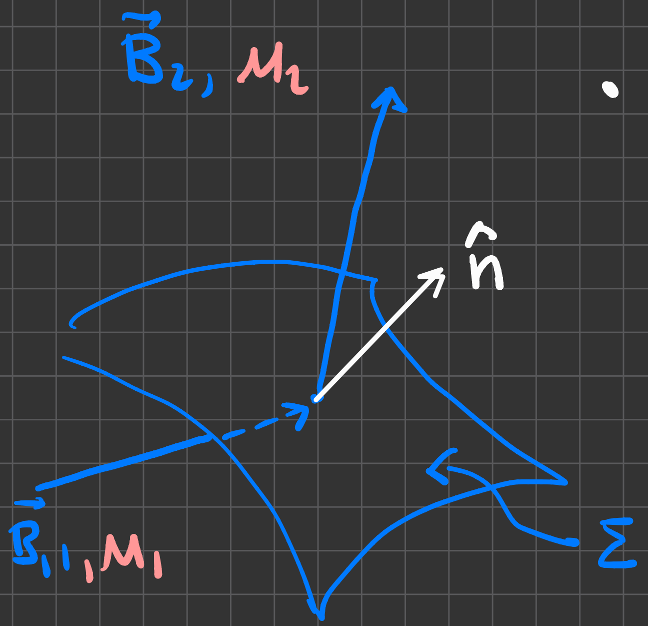
Again one can solve

$$\nabla \cdot \vec{B} = 0 \qquad \nabla \wedge \left(\frac{1}{\mu} \vec{B} \right) = \vec{J}$$

using the same tools as before but we need to be careful when there are boundaries.

Claim: • the component of $\vec{B}$ normal to Σ are continuous across Σ

$$(\vec{B}_2 - \vec{B}_1)\cdot \hat{n} = 0 \quad (\text{by } \nabla\cdot\vec{B} = 0)$$



• the component of $\frac{1}{\mu}\vec{B}$ tangent to Σ is not continuous and

$$\boxed{\left(\frac{1}{\mu_2}\vec{B}_2 - \frac{1}{\mu_1}\vec{B}_1\right)\wedge\hat{n} = I} \quad \left(\text{by } \nabla\wedge\frac{1}{\mu}\vec{B} = \vec{J}\right)$$

$\hat{n}$ = normal to Σ in the direction of $\vec{B}$ in region 2

Proof: exercise

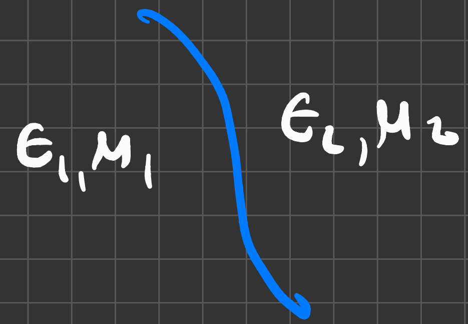
In summary:

$$\boxed{\begin{array}{ll} \nabla \cdot (\epsilon \vec{E}) = \rho & \nabla \wedge \vec{E} + \dfrac{\partial \vec{B}}{\partial t} = 0 \\[2em] \nabla \cdot \vec{B} = 0 & \nabla \wedge \left(\dfrac{1}{\mu} \vec{B}\right) - \epsilon \dfrac{\partial \vec{E}}{\partial t} = \vec{J}_s \end{array}}$$

$$\mu \epsilon = \frac{1}{v^2}$$

This lecture course:

for a given substance we assume ϵ, μ is constants but we allow for configurations with boundaries between materials with different values of ϵ, μ



[this is not true in general!
ϵ, μ depend on temperature, pressure, can vary across a substance, electromagnetic fields applied, etc ...]

Next: electromagnetic waves.