B7.2 ELECTROMAGNETISM

Chapter 5 : Electromagnetic waves (part 1)

Lecture 15

5 Electromagnetic waves

- Source free Maxwell's eqs admit wave solutions

properties of these solutions
electromagnetic spectrum

In a region of space where there a no sources ($\rho = 0$, $\vec{J} = 0$) Maxwell's equations are

$$\nabla \cdot \vec{E} = 0$$

(Gauss)

$$\nabla \wedge \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

Faraday

$$\nabla \cdot \vec{B} = 0$$

no monopoles

$$\nabla \wedge \vec{B} - \frac{1}{v^2} \frac{\partial \vec{E}}{\partial t} = 0$$

Ampere-Maxwell

in a medium characterized by (ϵ, μ), $v = \dfrac{1}{\sqrt{\mu \epsilon}}$

(in vacuum $v = c = 1/(\sqrt{\mu_0 \epsilon_0})$)

One can prove that

$$\Box \vec{E} = 0 \quad , \quad \Box \vec{B} = 0$$

In a region of space where there are no sources, the components of $\vec{E}$ & $\vec{B}$ satisfy the homogeneous wave equation with wave speed

$$v = 1/\sqrt{\epsilon\mu}$$

which in vacuum it is $c = \dfrac{1}{\sqrt{\mu_0 \epsilon_0}}$.

Proof:

$$0 = \nabla\wedge\Big(\underbrace{\nabla\wedge\vec E + \frac{\partial}{\partial t}\vec B}_{\text{Faraday}}\Big) = \nabla(\underbrace{\cancel{\nabla\cdot\vec E}}_{\text{Gauss}}) - \nabla^2\vec E + \frac{\partial}{\partial t}\underbrace{\nabla\wedge\vec B}_{\uparrow\text{ use Ampere}}$$

$$= -\nabla^2\vec E + \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\vec E = -\Box\vec E$$

Similarly $\quad 0 = \nabla\wedge\left(\nabla\wedge\vec B - \frac{1}{v^2}\frac{\partial^2\vec E}{\partial t^2}\right)$

together with $\quad \nabla\cdot\vec B = 0$

gives $\quad \Box\vec B = 0$

Another way to see this easily is wave eq satisfied by $(\Phi, \vec A)$ when there are no sources: $\Box\vec A = 0 \quad \Box\Phi = 0$

$\Rightarrow \quad \Box\vec B = \nabla\wedge\Box\vec A = 0 \qquad \Box\vec E = 0$ ✓

$\boxed{5.1}$ Plane electromagnetic waves

Consider

$$\Box \psi = 0 = -\frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} + \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}$$

in Cartesian coordinates

↳ find solutions of the wave equation

We will start with the simplest solutions

⇝ plane waves

Recall: the most general solution of the one dimensional wave equation

$$-\frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial x^2} = 0$$

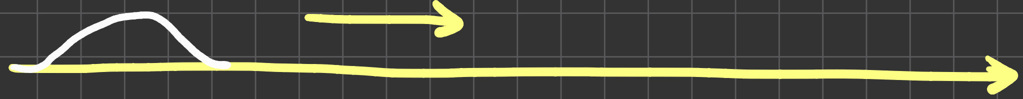
is

$$f(t,x) = f_R(x-vt) + f_L(x+vt)$$

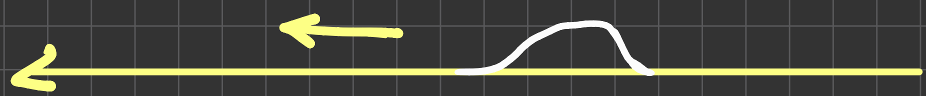
where f_R & f_L are arbitrary functions of one variable $x-vt$, respectively $x+vt$.

$f_R(x-vt)$ represents a wave propagating to the right with speed v

f_R = constant along $x-vt$ = const



$f_L(x+vt)$



Return now to the three dimensional wave eq

$$\Box\psi = 0 = -\frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} + \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}$$

in Cartesian coordinates

Try $\psi = f(\vec{k}\cdot\vec{r} - \omega t)$, $\vec{k}$ constant vector

(function of one variable) ω constant

Then $\frac{\partial\psi}{\partial x} = k_x f'$, $\frac{\partial^2\psi}{\partial x^2} = k_x^2 f''$

$\frac{\partial\psi}{\partial t} = -\omega f'$, $\frac{\partial^2\psi}{\partial t^2} = \omega^2 f''$

$\Rightarrow$ $\Box\psi = \left(-\frac{1}{v^2}\omega^2 + k^2\right) f''$ $\qquad k = |\vec{k}|$

Then $\psi = f(\vec{k}\cdot\vec{r} - \omega t)$ is a solution of $\Box\psi = 0$

iff $v^2 k^2 = \omega^2$

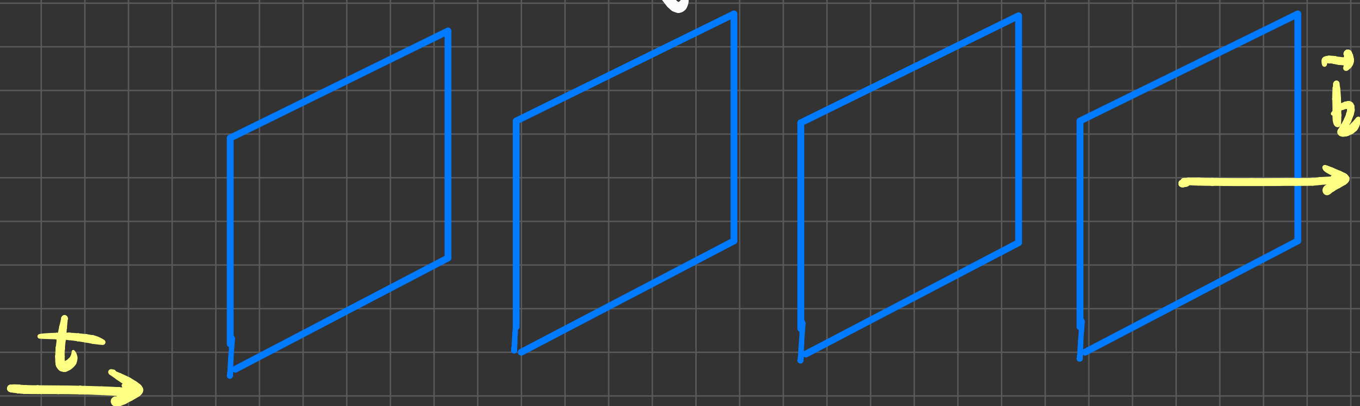
(true for any twice differentiable function f of one variable)

These solutions are called plane waves because:

$\psi =$ constant on $\vec{k}\cdot\vec{r} - \omega t =$ constant

↳ for each t this is the equation of a plane perpendicular to $\vec{k}$

so points of equal field value ψ lie on this plane



as time goes by this plane propagates in the direction of $\vec{k}$ at speed v

Harmonic waves or monocromatic plane waves of frequency ω are plane waves of the form

$$\psi = f(\vec{k}\cdot\vec{r} - \omega t) \sim e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

ψ oscillates with frequency ω

$k = |\vec{k}| = \frac{\omega}{v}$ wave number

$\uparrow$ # of cycles per unit of distance

Waves with frequency ω have wavelength

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{\omega} v$$

Note that:

1. These are of course complex solutions. However both the real and imaginary parts of $e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ are solutions of the wave equation, so one can take linear combinations of sin & cos

2. Fourier analysis:

A general solution of $\Box\psi = 0$ is a linear combination of monochromatic waves (series in terms of a complete set of orthonormal complex exponentials)

So far we have:

$$\begin{cases} \vec{E} = \vec{E}_0 \, e^{i(\vec{k}\cdot\vec{r} - \omega t)} \\ \vec{B} = \vec{B}_0 \, e^{i(\vec{k}\cdot\vec{r} - \omega t)} \end{cases} \qquad \begin{aligned} \vec{E}_0 &= E_0 \, \hat{e} \\ \vec{B}_0 &= B_0 \, \hat{b} \end{aligned} \qquad \begin{aligned} &\hat{e}, \hat{b} \text{ unit vectors} \\ &v^2 k^2 = \omega^2 \end{aligned}$$

but we have not exhausted Maxwell's equations

$$0 = \nabla \cdot \vec{E} = E_0 \, \nabla \cdot \left(e^{i\vec{k}\cdot\vec{r} - \omega t} \hat{e} \right) = E_0 \left(\nabla e^{i(\vec{k}\cdot\vec{r} - \omega t)} \cdot \hat{e} + e^{i(\vec{k}\cdot\vec{r} - \omega t)} \underbrace{\cancel{\nabla \cdot \hat{e}}}_{0} \right)$$

$$= E_0 \left(i\vec{k} \cdot \hat{e} \right) e^{i\vec{k}\cdot\vec{r} - \omega t} \qquad \therefore \quad \underline{\vec{k} \cdot \hat{e} = 0}$$

Similarly $\quad \nabla \cdot \vec{B} = 0 \quad$ iff $\quad \underline{\vec{k} \cdot \hat{b} = 0}$

Therefore $\vec{E}$ & $\vec{B}$ are perpendicular to $\vec{k}$

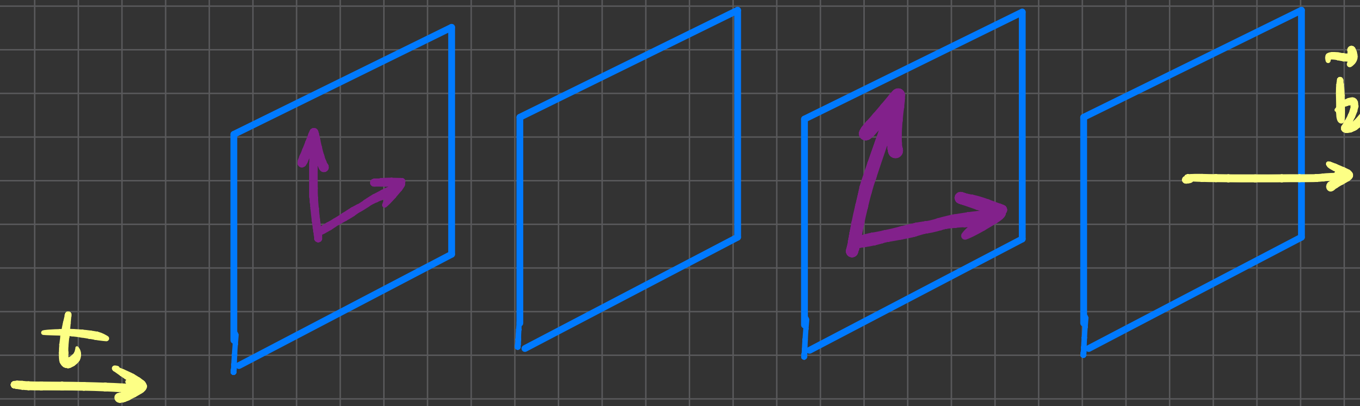
direction of propagation.

This is called a $\boxed{\text{transverse}}$ wave transverse vs longitudinal waves:

eg electromagnetic waves

eg sound waves

We have found that $\vec{E}$ & $\vec{B}$ oscillate on a plane transverse to the direction of propagation ($\vec{k}$)



Moreover: there is $\boxed{\text{no medium}}$, electromagnetic waves can propagate in vacuum at velocity c (with respect to any inertial reference frame!)

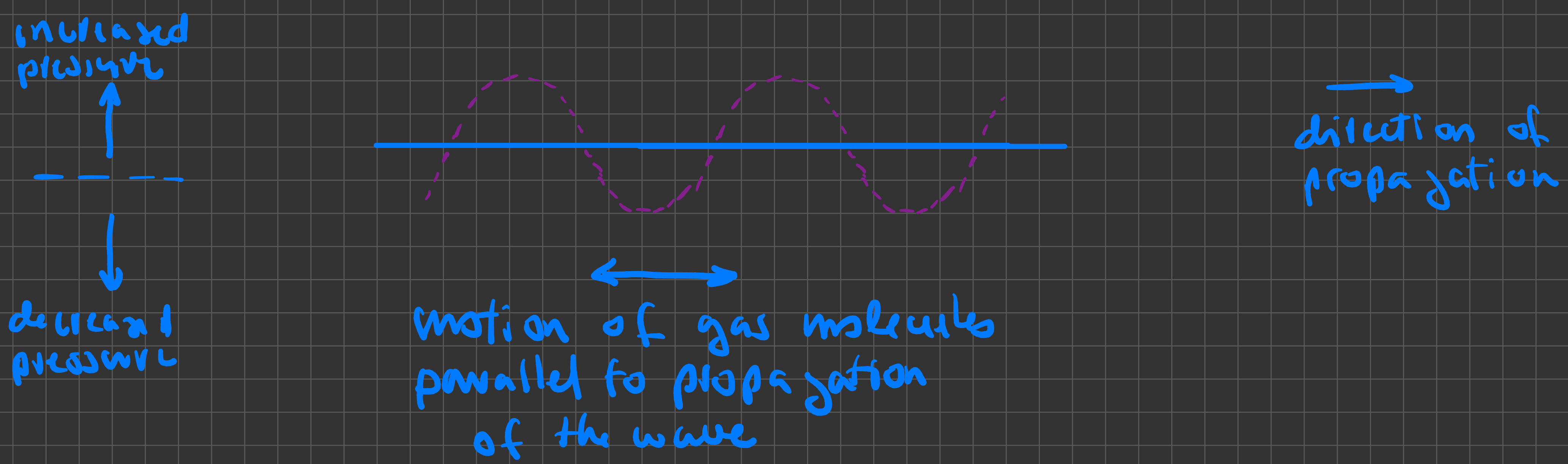
see Chapter 6

This is <u>unlike</u> sound waves !

Sound waves travelling in a gas (say air) are longitudinal waves (vibrations in the direction of travel)

They are mechanical waves → something vibrates

In fact sound waves travelling through air are pressure oscillatory variations in the gas caused by vibrations of molecules



So far we have

$$\vec{E} = E_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}\hat{e}, \quad \vec{B} = B_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}\hat{b}$$

with $\hat{e}\cdot\vec{k}=0 \quad \hat{b}\cdot\vec{k}=0$ and $\omega^2=(kv)^2$

For the other Maxwell's equations:

$$0 = \nabla\wedge\vec{E} + \frac{\partial}{\partial t}\vec{B} = E_0 \underbrace{\nabla\wedge(e^{i(\vec{k}\cdot\vec{r}-\omega t)}\hat{e})} + B_0\frac{\partial}{\partial t}e^{i(\vec{k}\cdot\vec{r}-\omega t)}\hat{b}$$

$$\nabla\wedge(e^{i(\vec{k}\cdot\vec{r}-\omega t)}\hat{e}) = e^{i(\vec{k}\cdot\vec{r}-\omega t)}\underbrace{\nabla\wedge\hat{e}}_{0} + (\nabla e^{i(\vec{k}\cdot\vec{r}-\omega t)})\wedge\hat{e}$$

$$= e^{i(\vec{k}\cdot\vec{r}-\omega t)}\, i\vec{k}\wedge\hat{e}$$

$$\Rightarrow \quad 0 = E_0\vec{k}\wedge\hat{e} - B_0\omega\hat{b}$$

Similarly from Ampère's law: $0 = \frac{\omega}{v^2}E_0\hat{e} + B_0(\vec{k}\wedge\hat{b})$

So

$$0 = E_0 \vec{k} \wedge \hat{e} - B_0 \omega \hat{b} \qquad (i)$$

$$0 = \frac{\omega}{v^2} E_0 \hat{e} + B_0 (\vec{k} \wedge \hat{b}) \qquad (ii)$$

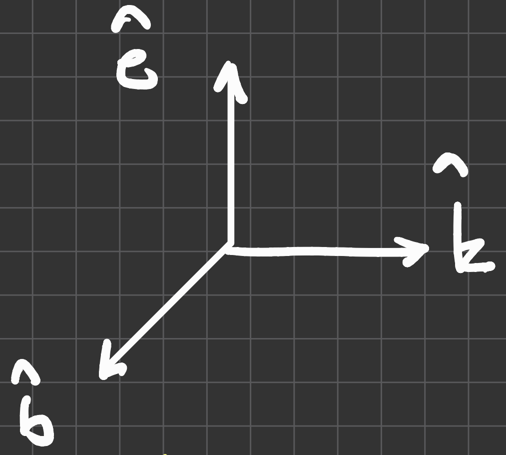
$\hat{e} \cdot (i)$ (or $\hat{b} \cdot (ii)$) : $\hat{e} \cdot \hat{b} = 0$

ie $\vec{E}$ & $\vec{B}$ are perpendicular to each other.

The vectors $\hat{e}, \hat{b}$ & $\hat{k}$ form a triad of vectors which are perpendicular to each other:

$$\hat{k} \wedge \hat{e} = \hat{b}$$
$$\hat{k} \wedge \hat{b} = -\hat{e}$$

where $\hat{k} = \frac{1}{k} \vec{k}$



(set of orthonormal vectors)

(i) $0 = E_0 \underbrace{\vec{k} \wedge \hat{e}}_{k\hat{b}} - B_0 \omega \hat{b} \quad \Rightarrow \quad E_0 k = B_0 \omega$

(ii) $0 = \frac{\omega}{v^2} E_0 \hat{e} + B_0 (\underbrace{\vec{k} \wedge \hat{b}}_{-k\hat{e}}) \quad \Rightarrow \quad \frac{\omega}{v^2} E_0 = k B_0$

[These are redundant as we already have $\omega^2 = (kv)^2$]

so $\quad E_0 = \frac{\omega}{k} B_0 = v B_0$

Summary Monocromatic plane wave solns

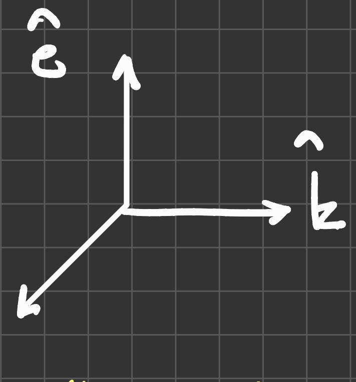
$$\vec{E} = E_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\,\hat{e}$$

$$v\vec{B} = \underbrace{\underbrace{vB_0}_{E_0}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}}_{|\hat{E}|}\,\underbrace{\hat{b}}_{\hat{k}\wedge\hat{e}} = \hat{k}\wedge\vec{E}\,, \qquad E_0 = vB_0$$

with $\omega = kv$ $\quad$ ($v = c$ in vacuum)

and

$$\hat{k}\wedge\hat{e} = \hat{b}$$
$$\hat{k}\wedge\hat{b} = -\hat{e}$$



(set of orthonormal vectors)

But what is propagating ??

energy and momentum

electromagnetic wave propagation corresponds to the transmission of energy and momentum.

Recall: (say in vacuum)

$$\mathcal{E}_e = \frac{\epsilon_0}{2}\left(|\vec{E}|^2 + c^2|\vec{B}|^2\right)$$

energy density of the electromagnetic field

For a monocromatic electromagnetic field

$$|\vec{E}|^2 = E_0^2 \qquad |\vec{B}|^2 = B_0^2 \qquad E_0 = cB_0$$

$$\mathcal{E}_e = \frac{\epsilon_0}{2}\left(E_0^2 + E_0^2\right) = \underline{\epsilon_0 E_0^2}$$

energy carried by the electromagnetic wave

Moreover: $\vec{P} = \frac{1}{\mu_0}(\vec{E} \wedge \vec{B})$ Poynting vector (momentum density)

$$\vec{P} = \frac{1}{\mu_0 c}\vec{E} \wedge (\hat{k} \wedge \vec{E}) = c\left(\epsilon_0 E_0^2\right)\hat{k} = c\,\mathcal{E}\,\hat{k}$$

$\vec{P}$: momentum density vector in the direction of $\hat{k}$ (direction in which wave is travelling) with $|\vec{P}| = c\,\mathcal{E}$ (expected from conservation of energy)

Remarks:

1. Consider a charged particle which is emitting electromagnetic waves (say a particle moving with velocity $\vec{v}(t)$)
Then this means it is emitting radiation and losing energy / momentum
(this is a problem for particle accelerators ...)

2. We talked about how you get hot when standing under the sun: you are absorbing radiation, that is energy is transfered from the electromagnetic waves to the molecules in your body

3 We radiate energy in the infrared
(thermal radiation)

wavelength
700×10^{-9} mt – 1 mm
(shorter than visible light)

In fact, everything emits radiation.

The frequency of the waves depends on the temperature of the body (which is determined by the atom's vibrations)

the hotter the atom, the faster it vibrates, the higher the frequency it emits

5.2 Polarization

↳ orientation of the oscillations of $\vec{E}$ and $\vec{B}$ on the plane perpendicular to the direction of travel $\vec{k}$

Choose a coordinate system with $\hat{k}$ along the z-axis.

We have

$$\vec{E} = E_0 e^{i(kz-\omega t)} \hat{e},$$

$$\vec{B} = B_0 e^{i(kz-\omega t)} \hat{b} = \frac{1}{v} \hat{k} \wedge \vec{E}$$

which are complex solutions of the wave equation

Consider linear combinations of the real/imaginary parts of these solutions

Observe what is happening from the plane $z=0$:

$$\vec{E} = \vec{e}_1 \cos\omega t + \vec{e}_2 \sin\omega t \qquad \vec{e}_1 = E_1 \hat{e}_1 \ , \text{ etc}$$

$$v\vec{B} = \hat{k} \wedge \vec{E} = \hat{k} \wedge \vec{e}_1 \cos\omega t + \hat{k} \wedge \hat{e}_2 \sin\omega t$$

where apriori $\vec{e}_1$ and $\vec{e}_2$ are independent.

This is a plane polarized wave: direction of the oscillations lie on the same plane

Case 1 linear polarization

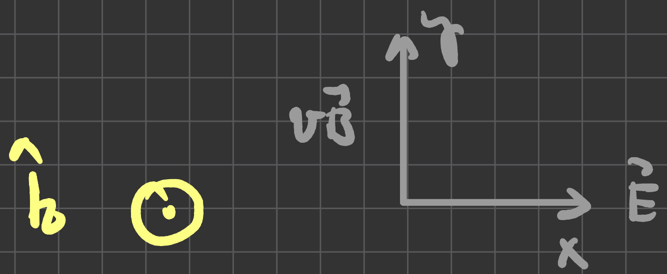
Let $\vec{e}_1$ parallel to $\vec{e}_2$ say $\hat{e}_1 = \hat{e}_2 = \hat{i}$

Then

$$\vec{E} = (E_1 \cos\omega t + E_2 \sin\omega t)\,\hat{i} = E\cos(\omega t + \delta)\,\hat{i}$$

some constant

$$v\vec{B} = \hat{k} \wedge \vec{E} = E\cos(\omega t + \delta)\,\hat{j}$$

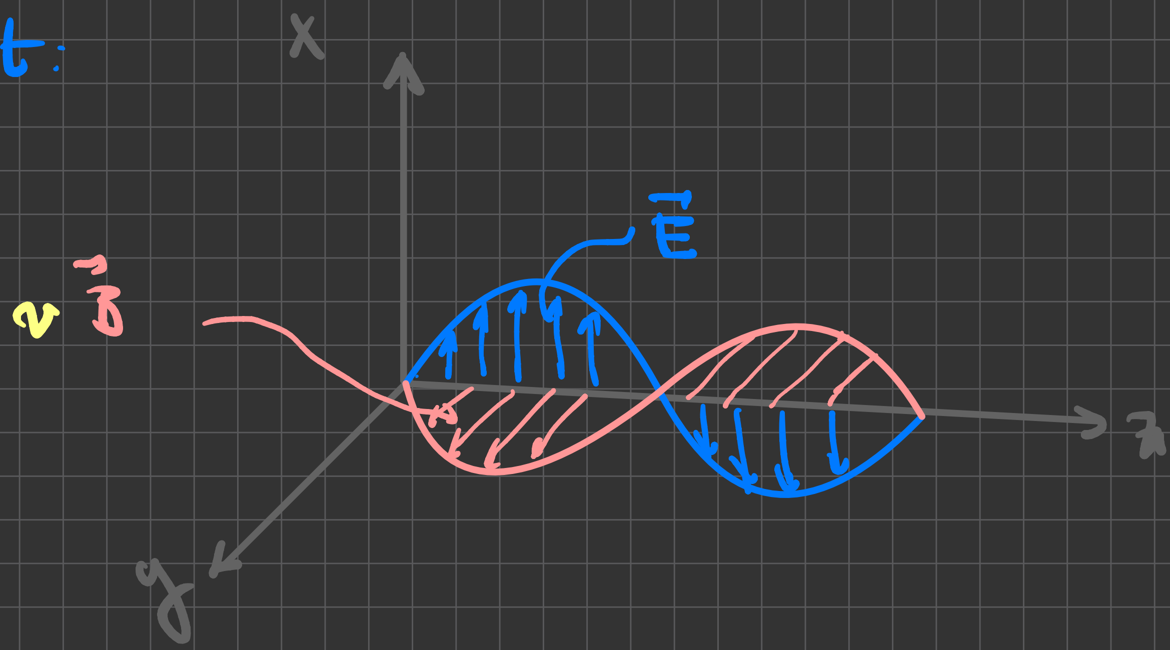


$\vec{E}$ and $\vec{B}$ are in phase, for example they vanish at the same points

This is called linear polarization because $\vec{E}$ and $\vec{B}$ oscillate along two perpendicular fixed lines

In our example $\vec{E}$ oscillates along the x-axis while $\vec{B}$ oscillates along the y-axis

For fixed t:



Case 2 Circular polarization

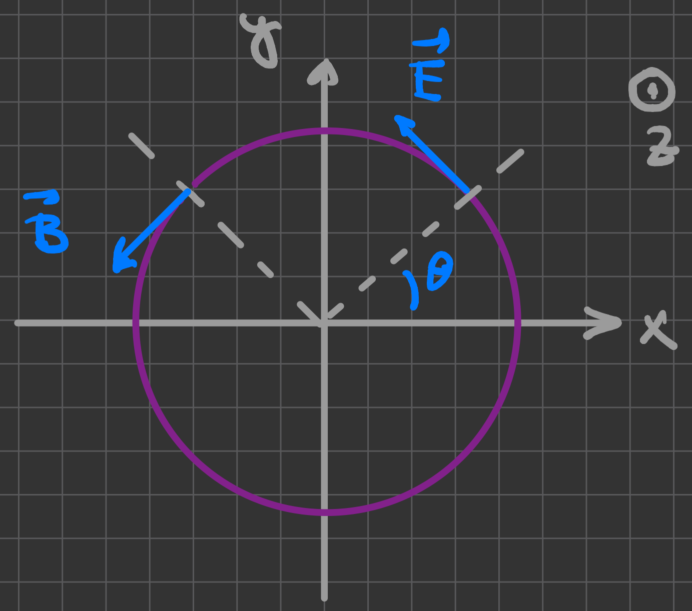
let $|\vec{e}_1| = |\vec{e}_2|$ and $\vec{e}_1 \cdot \vec{e}_2 = 0$

Say $\vec{e}_1 = E\,\hat{i}$, $\vec{e}_2 = E\,\hat{j}$

Then
$$\vec{E} = E(\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j})$$
$$v\vec{B} = \hat{k} \wedge \vec{E} = E(-\sin\omega t\,\hat{i} + \cos\omega t\,\hat{j})$$

let $\theta = \omega t$: $\hat{e}_\theta = \cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}$

$\vec{E} = E\,\hat{e}_\theta$ $\quad v\vec{B} = E\,\hat{e}_{\theta+\pi/2}$



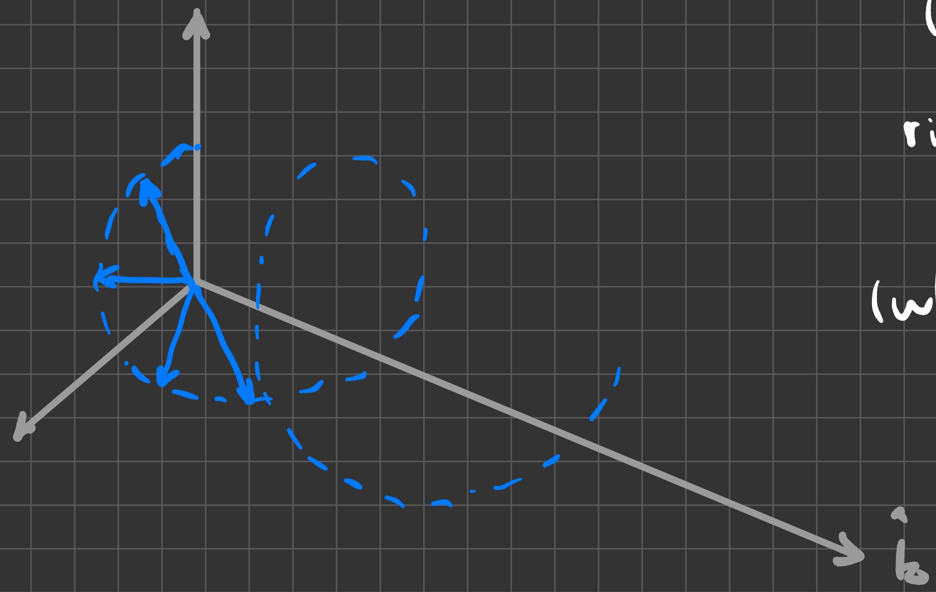
$\vec{E}, v\vec{B}$ tangent to a circle of radius E

As time goes by $\vec{E}$ & $\vec{B}$ rotate at a constant rate ω about the direction of propagation.

This is called a circularly polarized wave.

(Exercise: show that this is a superposition of two linearly polarized waves)

fixed t



left circularly polarized wave
→ anticlockwise rotation
(when $\vec{e}_1 \wedge \vec{e}_2$ is parallel to $\underline{k}$)

right circularly polarized wave
→ clockwise rotation
(when $\vec{e}_1 \wedge \vec{e}_2$ is anti-parallel to $\underline{k}$)

[can also have elliptical polarization]

General case → unpolarized light is a combination of waves with different polarization

Next: - reflection and refraction of electromagnetic waves → wave like behavior of solutions.

- electromagnetic spectrum.