B7.2 ELECTROMAGNETISM

Chapter 6 : Electromagnetism and special relativity

Lecture 17

(Last lecture!)

6 Electromagnetism & special relativity

The forces of nature as they were understood at the end of 19th C were the following:

Newtonian mechanics (~ 1670) | $\vec{F} = m\vec{a}$

Newton's law of gravitation (~1670) $\vec{F} = G\, mm' \frac{1}{r^3}\vec{r}$

Maxwell's theory of electromagnetism (~1864)

Einstein's theory of special relativity, proposed by Einstein in 1905, is a model for space time ($\mathbb{R}^4$) and its symmetries, derived from two axioms.

It resolved an inconsistency that appeared at the end of the 19th century between Newtonian mechanics & Maxwell's laws of electromagnetism

↳ the inconsistency was due to Newtonian conceptions of space time

Special relativity describes the motion of particles and bodies (Mechanics!) and replaces Newtonian mechanics. It

- reduces to Newtonian mechanics in the limit $v \ll c$
- is experimentally confirmed
- is good up to large scales when gravitational effects become important ⇝ general relativity
  (eg gravitational field of a black hole causes spacetime to curve)

Axiom 1 Principle of (special) relativity

all laws of physics must be consistent with special relativity

in particular:

- Maxwell's equations are (and they unify electric & magnetic forces)
- Newtonian mechanics isn't!
  - but they under the Principle of Galilean relativity instead.

moving constant velocity
relative to each other

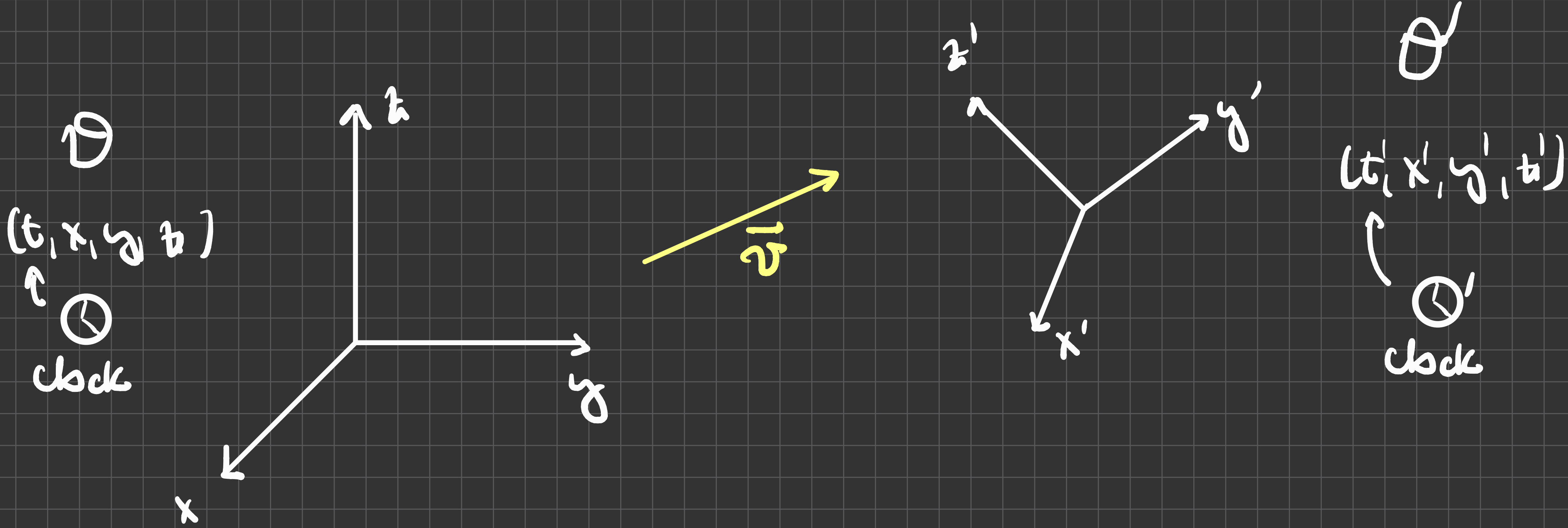
Consider the reference frame of two inertial observers $\mathcal{O}$ & $\mathcal{O}'$

$\mathcal{O}$ sets up a coordinate system to make measurements

(t, x, y, z)

$\mathcal{O}'$ sets up a coordinate system to make measurements

(t', x', y', z')



how are these IRFs related ?

$\mathcal{O}$ & $\mathcal{O}'$ develop laws of physics by making measurements in their own reference frames: how do these laws compare ?

Galileo says (Galilean relativity)

$$t' = t + t_0 \quad \Longrightarrow \quad \Delta t' = \Delta t$$

constant

time interval between two events is the same wrt Θ & Θ'

Galilean transformation

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = R \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \vec{v}\, t$$

rotation in $\mathbb{R}^3$

position of the origin of Θ at $\vec{v}\,t$ wrt Θ'

- Newtonian mechanics is invariant under Galilean transformations ✓

eg Θ would say $\vec{F} = m\vec{a}$

Θ' would say $\vec{F}' = m\vec{a}'$

- Lorentz says: Maxwell's equations are not invariant under Galilean transformations!

For example: the wave eq is not invariant under Galilean transformations

Suppose $f'(t', \vec{r}')$ satisfies the wave eq wrt $\mathcal{O}'$

$$\Box' f' = -\frac{1}{c^2}\frac{\partial^2 f'}{\partial t'^2} + \nabla'^2 f' = 0$$

Thus with respect to $\mathcal{O}$, $f(t, \vec{r}) = f'(t', \vec{r}')$ satisfies

$$\Box f - \frac{2}{c^2}\vec{v}\cdot\nabla\frac{\partial f}{\partial t} + \frac{1}{c^2}(\vec{v}\cdot\nabla)(\vec{v}\cdot\nabla f) = 0$$

Clearly a problem in electromagnetism.

Instead Maxwell's equations are invariant under affine linear transformations of spacetime $\mathbb{R}^4$

Lorentz transformation (see below)

$$\underbrace{\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix}}_{O'} = \underbrace{L}_{\text{constant } 4\times4 \text{ matrix}} \underbrace{\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}}_{O} + \underbrace{C}_{\text{constant 4-vector}}$$

Postulates of special relativity:

- Laws of physics are the same in all inertial reference frames (IRF)

  IRFs → frames moving with constant velocity relative to each other

  So for electromagnetism this means that forces on distributions of charges and currents are described by Maxwell's equations with ϵ_0, μ_0 in $\boxed{\text{any}}$ IRF

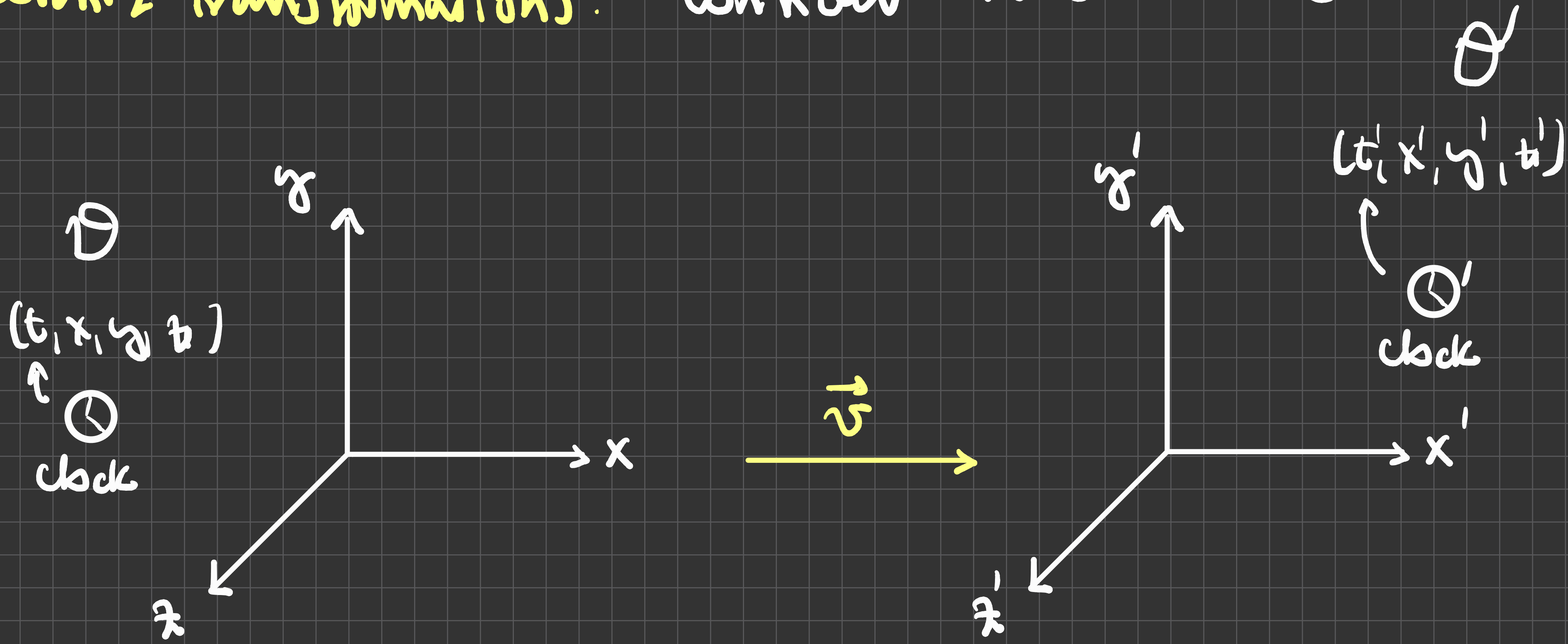
- The speed of light $c = (\epsilon_0 \mu_0)^{-1/2}$ is the same in all IRFs.

Einstein proved that Lorentz transformations follow from these postulates

- replace Newtonian mechanics by special relativity

- Maxwell's equations already invariant ✓
  In particular electromagnetic waves in vacuum move with velocity c in any IRF

Lorentz transformations: Consider two IRFs:



assume: at $t = t' = 0$ the origin of $\mathcal{O}$ coincides with the origin of $\mathcal{O}'$

and suppose that at this moment a flash light is sent.

This flash of light expands spherically with velocity c with respect to $\boxed{\text{both}}$ IRFs

With respect to $\mathcal{O}$: at a time t in their clock this sphere has equation

$$x^2 + y^2 + z^2 = (\underbrace{ct}_{})^2$$

distance travelled

With respect to $\mathcal{O}'$: at a time t' in their clock this sphere has equation

$$x'^2 + y'^2 + z'^2 = (\underbrace{ct'}_{})^2$$

distance travelled

Galileo would have said that these observations are related by

$$t' = t$$
$$x' = x - vt$$
$$y' = y$$
$$z' = z$$

Hence

$$\begin{array}{ccc} x'^2 + y'^2 + z'^2 & = & (x - vt)^2 + y^2 + z^2 \\ \| & & \| \\ (ct')^2 & = & (ct)^2 \end{array}$$

which is inconsistent with the predictions of electromagnetism

On the other hand Lorentz, Poincaré, Einstein would have said that the linear transformation

$$X' = LX \quad , \quad X' = \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} \quad X = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \quad (\text{on } \mathbb{R}^4)$$

which leaves the equation

$$-(ct)^2 + x^2 + y^2 + z^2 = X^T \eta X, \quad \eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$= 0$

invariant is :

$$ct' = \frac{1}{\sqrt{1-v^2/c^2}}\left(ct - \frac{v}{c}x\right)$$

$$x' = \frac{1}{\sqrt{1-v^2/c^2}}(x - vt)$$

$$y' = y$$

$$z' = z$$

ie
$$L = \begin{pmatrix} \gamma & -\gamma\frac{v}{c} & 0 & 0 \\ -\frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}} > 1$$

L is not well defined when $v > c$

no real valued transformations, when $v > c$ between O & O'

[Lorentz found this from Maxwell's eqs]

Sketch of the proof:

let $X' = LX$ for some 4×4 matrix L

$$\underset{0}{\underset{\parallel}{X'^T \eta X'}} = X^T \underbrace{(L^T \eta L)}_{} X = \underset{0}{\underset{\parallel}{X^T \eta X}}$$

Hence $L^T \eta L = \mu \eta$ for some $\mu \in \mathbb{R}$

[$\mu = 1$ because $\underset{\gamma}{\underset{\parallel}{(L^{-1})^0{}_0}} = \mu^{-1} \underset{\gamma}{\underset{\parallel}{L^0{}_0}}$] non [illegible]

Then L 4×4 matrices having η invariant $L^T \eta L = \eta$

[($L \in SO(1,3)$)]

Remarks:

1) For $v \ll c$ $\qquad c = 300\,000$ km/sec

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = 1 + \frac{1}{2}\left(\frac{v}{c}\right)^2 + \cdots$$

Then: Lorentz transformation $\longrightarrow$ Galilean transformation

compare for example: a car with $v = 100$ km/hr

$$v = 100 \,\frac{\text{km}}{\text{hr}} = \frac{1}{36}\,\text{km/sec} \Rightarrow \left(\frac{v}{c}\right)^2 \sim (10^{-7})^2$$

2

- $\mathcal{O}$ measures the length of a stationary bar and finds ℓ

$\mathcal{O}'$ says ℓ'

One can show : $\ell' = \frac{1}{\gamma}\ell$ so $\ell' < \ell$

- $\mathcal{O}$ says that Δt is the time interval between two events at the same place in space

$\mathcal{O}'$ says $\Delta t'$

Then : $\Delta t' = \gamma \Delta t$ ie "$\mathcal{O}'$ is <u>slower</u> than $\mathcal{O}$"

3 Schrödinger equation in quantum mechanics

Let $\Psi(t, \vec{r})$, respectively $\Psi'(t', \vec{r}')$, be the wave function of a particle relative to $\mathcal{O}$ and respectively $\mathcal{O}'$. Schrödinger's equation

$$-\frac{\hbar}{2m} \nabla^2 \Psi + V\Psi = i\hbar \frac{\partial \Psi}{\partial t}$$

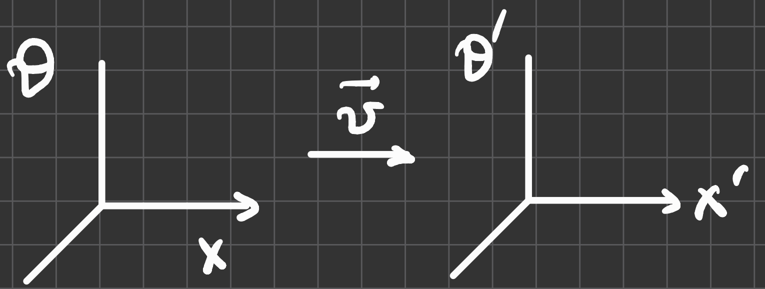
is invariant under Galilean transformations as long as V is and the wave function changes by a fase (so $|\Psi|^2 = |\Psi'|^2$)

Need to correct this $\rightsquigarrow$ relativistic quantum mechanics

Jackson

Exercise: Under a Lorentz transformation

$$X' = LX \, , \qquad L = \begin{pmatrix} \gamma & -\gamma\frac{v}{c} & 0 & 0 \\ -\frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \gamma = \frac{1}{\sqrt{1-v^2/c^2}} > 1$$



$$E_x' = E_x \qquad B_x' = B_x$$

$$E_y' = \gamma(E_y - vB_z) \qquad B_y' = \gamma\left(B_y + \frac{v}{c^2}E_z\right)$$

$$E_z' = \gamma(E_z + vB_y) \qquad B_z' = \gamma\left(B_z - \frac{v}{c^2}E_y\right)$$

component along relative motion are the same

Then one can show that if $\vec{E}$ & $\vec{B}$ satisfy Maxwell's equations with respect to $\mathcal{O}$ then so do $\vec{E}'$ & $\vec{B}'$ with respect to $\mathcal{O}'$

$$\nabla' \cdot \vec{E}' = \frac{1}{\epsilon_0} \rho' \quad \cdots \qquad \nabla \cdot \vec{B}' = 0 \qquad \text{etc.}$$

$$\nabla \wedge \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

Example Suppose that wrt $\mathcal{O}$ a charge q is moving with constant velocity $\vec{v} = v\,\hat{i}$

so $\rho(t, \vec{r}) = q \; \delta(\vec{r} - \vec{r}_0(t))$

position of q at t
(eq of trajectory)

$\vec{J}(t, \vec{r}) = q\,\vec{v} \quad \delta(\vec{r} - \vec{r}_0(t))$, $\quad \vec{v} = \frac{d}{dt}\vec{r}_0(t)$

constant

Chapter 4:

$$\Phi(t, \vec{r}) = \frac{q}{4\pi\epsilon_0} \int dt' \frac{1}{|\vec{r} - \vec{r}_0(t')|} \, \delta(t - t' - \tfrac{1}{c}|\vec{r} - \vec{r}(t')|)$$

$\vec{A}(t, \vec{r}) = \cdots$

etc.

Alternatively, in the particles' reference frame O'

$$\vec{E}' = \frac{q}{4\pi\epsilon_0}\frac{1}{r'^3}\vec{r}' \quad \& \quad \vec{B}' = 0$$

using Lorentz transformations

$$\vec{r}' = \gamma(x - vt)\hat{i} + y\hat{j} + z\hat{k}$$

$$r'^2 = \gamma^2(x-vt)^2 + y^2 + z^2$$

$$0 = B'_x = B_x \qquad B_x = 0$$

$$0 = B'_y = \gamma\left(B_y + \frac{v}{c^2} E_z\right) \Longrightarrow B_y = -\frac{v}{c^2} E_z \Longleftrightarrow \vec{B} = -\frac{v}{c^2} \hat{i} \wedge \vec{E}$$

$$0 = B'_z = \gamma\left(B_z - \frac{v}{c^2} E_y\right) \qquad B_z = \frac{v}{c^2} E_y$$

$$E'_x = E_x \Longrightarrow E_x = E'_x$$

$$E'_y = \gamma(E_y - v B_z) \Longrightarrow E'_y = \gamma E_y \left(1 - \frac{v^2}{c^2}\right) = \gamma^{-1} E_y \Rightarrow E_y = \gamma E'_y$$

$$E'_z = \gamma(E_z + v B_y) \Longrightarrow E'_z = \gamma E_z \left(1 - \frac{v^2}{c^2}\right) = \gamma^{-1} E_z \Rightarrow E_z = \gamma E'_z$$

ie $\vec{E} = E'_x \hat{i} + \gamma E'_y \hat{j} + \gamma E'_z \hat{k}$

Using $\vec{E}' = \frac{q}{4\pi\epsilon_0} \frac{1}{r'^3} \vec{r}'$

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{1}{r'^3} \left(x'\hat{i} + \gamma y'\hat{j} + \gamma z'\hat{k}\right)$$

$$= \frac{q}{4\pi\epsilon_0} \frac{\gamma}{r'^3} \left((x-vt)\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{\gamma}{r'^3} \left(\vec{r} - \vec{v}t\right) \qquad \vec{B} = -\frac{v}{c^2} \hat{i} \wedge \vec{E}$$

where

$$r'^2 = \gamma^2 (x-vt)^2 + y^2 + z^2$$

End of the lecture course

Thank you !