Numerical Solution of Differential Equations II. QS 1 *(HT 2021)*

1. By (artificially) employing analytic solution methods for initial value problems (but not for the BVP) solve

$$u'' - 2u' + 2u = 2, \quad u(0) = 1, \quad u(\frac{\pi}{2}) = 2$$

by the shooting method for linear problems. (You may find that selection of initial slopes $u_0'(0) = 0$ and $u_1'(0) = 1$ where u_0 solves the inhomogeneous IVP and u_1 solves the homogeneous IVP makes the calculation a little easier, although any values should work!)

2. As Qn 1 above; why would there be a difficulty if the right hand boundary condition was set to $u(\pi) = 2$? What solution(s) exist in this case?

3. Show that solving

$$u'' = 2u' - 2u + 2, \quad u(0) = 1, \quad u(\frac{\pi}{2}) = 2$$

by the general method for nonlinear equations gives the exact initial slope which gives the solution of the BVP at the first Newton iteration when $u'(0) = s = 0$ is chosen. Is this true for any other values of s?

4. Use the shooting method to solve the BVP

$$u'' = 100u, \quad u(0) = 1, \quad u(3) = \epsilon + \cosh 30.$$

If the linearly independent solutions

$$u_0(x) = \cosh 10x, \quad u_1(x) = \frac{1}{10}\sinh 10x$$

are employed, show that

$$\mu = 10\epsilon / \sinh 30.$$

Explain the difficulty in carrying out the numerical solution for small $|\epsilon|$.

5. For the problem

$$(pu')' + qu = f, \quad u(a) = \alpha, \quad u(b) = \beta,$$

approximated by

$$\frac{p(x_{j+\frac{1}{2}})u_{j+1} - \left[p(x_{j+\frac{1}{2}}) + p(x_{j-\frac{1}{2}})\right] u_j + p(x_{j-\frac{1}{2}})u_{j-1}}{h^2} + q(x_j)u_j = f(x_j),$$

$$j = 1, \ldots, n, \quad u_0 = \alpha, \quad u_{n+1} = \beta,$$

show that the local truncation error is $\mathcal{O}(h^2)$ provided $u(x)$ and $p(x)$ have four derivatives.
[Note: $p(x_{j\pm\frac{1}{2}}) = p(x_j) \pm \frac{h}{2}p'(x_j) + \frac{1}{8}h^2p''(x_j) \pm \frac{h^3}{48}p'''(x_j) + \mathcal{O}(h^4)$.]

6. If A is SCDD ($|a_{jj}| > \sum_{\substack{i=1\\i\neq j}}^{n} |a_{ij}|$ each j) and after the first step of Gaussian Elimination it is reduced to

$$\begin{bmatrix} a_{11} & a_{12} \ldots a_{1n} \\ 0 & B \end{bmatrix},$$

prove that B is SCDD. Deduce that for SCDD A, all multipliers used in Gaussian Elimination have absolute value less than 1.

7. The Poisson problem $\nabla^2 u = f$ is to be solved on a spherically symmetric region $a \leq r \leq b$ leading to the problem of finding $u(r)$ which satisfies:

$$\frac{1}{r^2}\frac{d}{dr}(r^2\frac{du}{dr}) = f(r), \quad a < r < b, \quad u(a) = \alpha, \quad u(b) = \beta.$$

Write down the finite difference approximation as in Qu. 5 above (i.e. $(pu')' = r^2 f$ with $r^2 = p$) on a regular grid of size h : check that the resulting matrix is symmetric tridiagonal.

8. In Qu. 7 above, if the PDE is now $\nabla^2 u - u = f$, prove that the matrix resulting is SRDD for any finite h (and hence symmetric $\Longrightarrow$ SCDD).