Numerical Solution of Differential Equations II. QS 2 *(HT 2021)*

1. Show that approximation of

$$(pu')' = pu'' + p'u' = f, \quad u(a) = \alpha, \quad u(b) = \beta,$$

by the finite difference approximation

$$p(x_j)\frac{U_{j+1} - 2U_j + U_{j-1}}{h^2} + p'(x_j)\frac{U_{j+1} - U_{j-1}}{2h} = f(x_j), \quad U_0 = \alpha, \quad U_{n+1} = \beta$$

yields a tridiagonal but <u>not</u> symmetric matrix.

2. The equation

$$u'' - au' - bu = g, \quad 0 \leq x \leq 1,$$

(where a, $b > 0$, g are suitably differential functions), is discretised using central differences on a uniform mesh of size h.

Show that the coefficient matrix for the discrete problem is diagonally dominant provided

$$h < \frac{2}{\max|a|}.$$

3. Let $\{a_1, a_2, \ldots, a_m\}$ be real numbers and the tridiagonal matrix P

$$P = \begin{pmatrix} a_1 + a_2 & -a_2 & 0 & \ldots & 0 & 0 \\ -a_2 & a_2 + a_3 & -a_3 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -a_{m-1} & a_{m-1} + a_m \end{pmatrix}.$$

Show that provided

$$\min\{a_r\} > 0$$

then P is positive definite.

4. The transverse displacement of a beam, $u(x)$, satisfies

$$u^{(iv)} = f(x), \quad 0 \leq x \leq 1,$$

with boundary conditions

$$u(0) = u''(0) = u(1) = u''(1) = 0.$$

(a) Discretise the ODE directly on a uniform mesh of size $h = 1/(n+1)$, (n a positive integer) using central differences. Consider carefully how you will treat the first and last interior point. What would you do if the boundary condition $u''(0) = 0$ was changed to $u'(0) = 0$?

(b) Write the equation as a system:

$$\begin{aligned} u'' &= w, \\ w'' &= f. \end{aligned}$$

Discretise this system using finite differences. In each case write down the coefficient matrix for the system of linear equations which would have to be solved to yield a solution.

5. By use of Taylor Series, show that the local truncation error of the finite difference scheme

$$\frac{-\frac{1}{12}U_{j+2} + \frac{4}{3}U_{j+1} - \frac{5}{2}U_j + \frac{4}{3}U_{j-1} - \frac{1}{12}U_{j-2}}{h^2} = f(x_j)$$

for the problem $u'' = f$ satisfies $\tau_j = \mathcal{O}(h^4)$.

6. Show that for each $r, s = 1, \ldots, n$, the vector

$$v^{rs} = (v^{rs}_{11}, v^{rs}_{12}, \ldots, v^{rs}_{1n}; v^{rs}_{21}, v^{rs}_{22}, \ldots, v^{rs}_{2n}; \ldots; v^{rs}_{n1}, v^{rs}_{n2}, \ldots, v^{rs}_{nn})^T$$

with $v^{rs}_{jk} = \sin \frac{jr\pi}{n+1} \sin \frac{ks\pi}{n+1}$ is an eigenvector of the 5-point finite difference matrix with corresponding eigenvalue $\frac{1}{h^2}\left[4 - 2\cos\frac{r\pi}{n+1} - 2\cos\frac{s\pi}{n+1}\right]$.

7. If $A \in \mathbb{R}^{m \times m}$ is symmetric and reducible, prove that its eigenvectors fall into (at least) 2 natural sets, the span of which are two orthogonal subspaces (i.e. if the 2 subspaces are U, V, then for all $u \in U$, $v \in V$, $u^T v = 0$).

Give an example of a singular but diagonally dominant matrix which is strictly diagonally dominant in at least one row.