Numerical Solution of Differential Equations II. QS 3 *(HT 2021)*

1. Suppose Ω is the right-angled isosceles triangle

$$\{(x,y)\ :\ 0 \le x \le 1,\ \ 0 \le y \le 1-x\}$$

and $\delta\Omega$ its boundary. Use a regular grid on this domain with $x_0 = 0$, $x_j = jh$, $(n+1)h = 1$, $y_0 = 0$, $y_k = kh$ and apply the 5-point finite difference formula to approximate the solution of $-\nabla^2 u = f$ on Ω with $u = 0$ on $\delta\Omega$. How many rows of the coefficient matrix have only 2 non-zero entries? How many have 3, 4 non-zero entries? What is the structure of the coefficient matrix with Lexicographic ordering? (This is not so easy as for a square! Do it for $n = 5$ if you prefer.) Is this matrix Irreducibly Diagonal Dominant?

2. Verify that the matrix obtained for central difference approximation of

$$u'' + qu = f \ \text{ on } \ [a,b]\,,\quad u(a) = \alpha,\quad u'(b) = \beta,$$

where $q \in \mathbb{R}$ is a constant, is Irreducibly Diagonally Dominant for every (small) mesh size h if and only if $q \le 0$.

3. If

$$L_h U_{j,k} = \frac{1}{h^2}\left(4U_{j,k} - U_{j+1,k} - U_{j-1,k} - U_{j,k+1} - U_{j,k-1}\right),$$

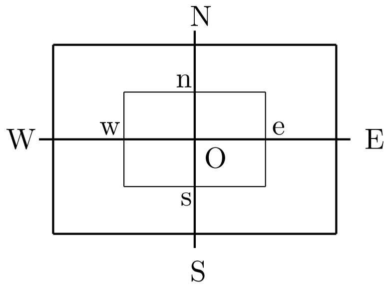
show that $L_h \Psi_{j,k} = -4$, where $\Psi_{j,k} = (x_j - \frac{1}{2})^2 + (y_k - \frac{1}{2})^2$. Hence, by using the minimum principle on $\phi_{j,k} = e_{j,k} - \frac{1}{4}\tau\Psi_{j,k}$, prove that $-\frac{\tau}{8} \le e_{j,k},\ \ j,k = 1,\ldots,n$.

4. Suppose the rectangular region $\Omega = [0, H] \times [0, L] \in \mathbb{R}^2$ is covered with a mesh of n points in each of the x and y coordinate directions where $h = \frac{H}{n+1}$ is the mesh size in the x direction and $l = \frac{L}{n+1}$ is the mesh size in the y direction.

If the problem $\frac{\partial}{\partial x}\left(p\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(p\frac{\partial u}{\partial y}\right) = f$ on Ω with u given on $\partial\Omega$ is to be solved with p a continuous function which does not change sign in $\partial\Omega$, derive the formula

$$\frac{h}{l}p(s)U_S+\frac{l}{h}p(e)U_E+\frac{h}{l}p(n)U_N+\frac{l}{h}p(w)U_W-\left(\frac{h}{l}p(s)+\frac{l}{h}p(e)+\frac{h}{l}p(n)+\frac{l}{h}p(w)\right)U_0 = hlf(0)$$

from applying the Divergence Theorem to the indicated rectangular area and using simple approximation for the line integrals which result.



Verify that in the special case $p = -1$ and $h = l$ this is the usual 5-point formula. For the non-special case, is the matrix symmetric?

5. Consider the Neumann Problem for the Laplace Equation

$$-\nabla^2 u = 0 \ \text{ in } \ (0,1)\times(0,1) = \Omega, \quad \frac{\partial u}{\partial n} = 0 \ \text{ on } \ \partial\Omega.$$

(a) If the boundary condition is approximated by central differences, eg. $\frac{U_{-1,k}-U_{1,k}}{2h} = 0$ on the left side and correspondingly on the other 3 sides, and the fictitious values (e.g. $U_{-1,k}$) are eliminated from the standard 5-point finite difference stencil which is used in the interior and up to the boundary, write down the structure of the resulting $(n+2)^2\times(n+2)^2$ matrix, A, which results under the usual lexicographic ordering.

(b) Is A irreducibly diagonally dominant?

(c) Is A non-singular?

(d) Show that the vector v^{rs}, $r, s = 0, \ldots, n+1$ with entries

$$v^{rs}_{jk} = \cos\frac{jr\pi}{n+1}\cos\frac{ks\pi}{n+1}$$

is and eigenvector of A : find the corresponding eigenvalue.

(e) Comment on v^{00}.

(f) Show that fixing the value of any particular U_{jk} on $\partial\Omega$ (which corresponds to applying a Dirichlet boundary condition at one point) will make the finite difference solution unique.