Numerical Solution of Differential Equations II. QS 4 *(HT 2021)*

1. Calculate the local truncation error for the Lax-Wendroff finite difference scheme for the first order wave equation. What is the CFL stability criteria for this scheme.

2. By writing the scheme in the form

$$U_j^{n+1} = \mathcal{H}_k U^n,$$

prove that the first order upwind scheme is stable for the first order wave equation provided

$$\left| \frac{ak}{h} \right| \leq 1.$$

You should use the discrete norm defined by

$$\|U^n\| = h \sum_j |U_j^n|.$$

3. Derive the first order upwind scheme for the initial value problem

$$\frac{\partial \mathbf{u}}{\partial t} + \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix} \frac{\partial \mathbf{u}}{\partial x} = 0, \quad \mathbf{u}(x,0) = \mathbf{u}_0(x).$$

What is the CFL stability criterion in this case?

4. For the nonlinear conservation law

$$u_t + f(u)_x = 0$$

find a numerical flux for the Lax-Wendroff finite difference scheme so that it can be written in conservation form. Show also that the Lax Wendroff scheme for the first order wave equation can be written as

$$U_j^{n+1} = U_j^n - \nu(U_j^n - U_{j-1}^n) - \frac{1}{2}\nu(1-\nu)(U_{j+1}^n - 2U_j^n + U_{j-1}^n).$$

5. Verify that the Lax-Freidrich's scheme is a TVD scheme for the first order wave equation when the CFL criteria are satisfied, but that the Lax Wendroff scheme does not satisfy the sufficient conditions for a TVD scheme except when $\nu = 0, \pm 1$.