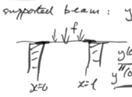
2 point BVPs.

General form $u'' = g(x, u, u')$

on $(a, b) \subset \mathbb{R}$

with $u(a) = \alpha$, $u(b) = \beta$

Example: deflection of a freely-supported beam: $y(x)$



$y(0) = 0$, $y(l) = 0$
$y''(0) = 0$, $y''(l) = 0$

and $EI \dfrac{d^4 y}{dx^4} = f(x)$

Young's Modulus (material parameter) — E

relates to beam cross section — I

Write $y'' = z$

$$z'' = \frac{1}{EI} f(x)$$

$$\Leftrightarrow u = \begin{pmatrix} y \\ z \end{pmatrix}, \quad g(x, u, u') = \begin{pmatrix} z \\ \frac{1}{EI} f \end{pmatrix}$$

$$u(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad u(l) = 0$$

Existence/Uniqueness is not at all straight forward

e.g. simplest linear problem

$$u'' + u = 0$$

$\Rightarrow u = A\cos x + B\sin x$

(i) if $u(0) = 0$, $u(\frac{\pi}{2}) = 1 \Rightarrow u = \sin x$, uniquely

(ii) $u(0) = 0$, $u(\pi) = 0 \Rightarrow u = B\sin x$, any B

(iii) $u(0) = 0$, $u(\pi) = 1 \Rightarrow$ no solution

Numerical Methods:

3 main types
- Shooting Methods
- Finite Difference Methods
- Expansion (Galerkin / Collocation) Methods

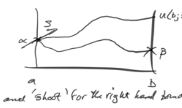
Part C Finite Element course

Shooting Methods: based on use of any method for an *initial* value problem for ODEs:

Idea: $u'' = g(x, u, u')$ on (a, b)

$u(a) = \alpha$, $[u(b) = \beta]$

'replace' by initial condition $u'(a) = s$



and 'shoot' for the right hand boundary.

Write solution of IVP as $u(x; s)$: want to find s so that

$$G(s) := u(b; s) - \beta = 0$$

$\Rightarrow$ problem reduces to solving the nonlinear equation $G(s) = 0$ where evaluation of G for any choice of s requires the solution of an IVP.

Could use bisection method;

find s_0, s_1 so that β is between $u(b; s_0)$ and $u(b; s_1)$

(*) set $s_2 = \dfrac{s_0 + s_1}{2}$ and then evaluate $u(b; s_2)$

if β between $u(b; s_0)$ and $u(b; s_2)$ set $s_1 \leftarrow s_2$

else β between $u(b; s_1)$ and $u(b; s_2)$ set $s_0 \leftarrow s_2$

and continue from (*) until convergence to desired accuracy

or could use any other iteration for nonlinear equations: in particular **Newton iteration**: for initial guess s_0 compute for $r = 0, 1, 2, \ldots$

$$s_{r+1} = s_r - G(s_r)/G'(s_r)$$

$$= s_r - \frac{[u(b; s_r) - \beta]}{G'(s_r)}$$

Note $G'(s) = \dfrac{\partial}{\partial s}(u(b; s))$

but $u'' = g(x, u, u')$

so $\dfrac{\partial}{\partial s} u'' = \dfrac{\partial g}{\partial u}\dfrac{\partial u}{\partial s} + \dfrac{\partial g}{\partial u'}\dfrac{\partial u'}{\partial s}$

and if we write $v(x; s) = \dfrac{\partial}{\partial s}(u(x; s))$

$$v'' = \frac{\partial g}{\partial u} v + \frac{\partial g}{\partial u'} v'$$

with $u(a; s) = \alpha \Rightarrow v(a; s) = 0$

$u'(a; s) = s \Rightarrow v'(a; s) = 1$

which can be solved by the IVP solver to give $v(x; s)$

So Newton's Method is

$$s_{r+1} = s_r - \frac{u(b; s_r) - \beta}{v(b, s_r)}$$

which requires the solution of 2 different IVPs at each iteration.

Linear problem

$$u'' = a(x)u + b(x)u' + c(x) \quad (+)$$

are easier because we can use the principle of superposition:

For some s_0 let u_0 solve

$$u_0'' = a(x)u_0 + b(x)u_0' + c(x)$$

with $u_0(a) = \alpha$, $u_0'(a) = s_0$

For some s_1 $(0 \neq s_1 \neq s_0)$ let u_1 solve

$$u_1'' = a(x)u_1 + b(x)u_1' \quad [\text{no } c!]$$

with $u_1(a) = 0$, $u_1'(a) = s_1$

then for any $\mu \in \mathbb{R}$, $u_0 + \mu u_1$ solves the linear ODE (+)

with $u_0(a) + \mu u_1(a) = \alpha$

and if μ is chosen so that

$$u_0(b) + \mu u_1(b) = \beta$$

i.e. $\mu = \dfrac{\beta - u_0(b)}{u_1(b)}$

then $u_0(x) + \mu u_1(x)$ satisfies the r.h. boundary condition also and so it is <u>the</u> solution of the BVP.