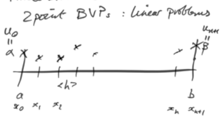
Finite Difference Method for 2point BVPs : linear problems



e.g. $u'' + pu' + qu = f$ in (a,b)

associate u_j with $u(x_j) = u(a+jh)$,

$$\boxed{h = \frac{b-a}{n+1}}$$

and approximate using finite differences
: using Taylor's Theorem

$$u(x_j \pm h) = u \pm hu' + \frac{h^2}{2}u'' \pm \frac{h^3}{6}u''' + \frac{h^4}{24}u^{iv}(\xi^{\pm})$$

some $\xi^+ \in (x_j, x_{j+1})$
$\xi^- \in (x_{j-1}, x_j)$

where $u = u(x_j)$, $u' = u'(x_j)$ etc.

So $u(x_{j+1}) - 2u(x_j) + u(x_{j-1})$

$$= h^2 u''(x_j) + \frac{h^4}{24}\left[u^{iv}(\xi^+) + u^{iv}(\xi^-)\right]$$

Thus we approximate

$$u'' \to \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2}$$

and similarly

$$u' \to \frac{u_{j+1} - u_{j-1}}{2h}.$$

Thus we approximate the solution of the ODE by the values $u_1, u_2, \dots, u_n$ which satisfy the n equations

$$\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} + p(x_j)\frac{u_{j+1} - u_{j-1}}{2h} + q(x_j)u_j = f(x_j)$$

$j = 1, 2, \dots, n$

with $u_0 = \alpha$, $u_{n+1} = \beta$.

This is a linear system of equations with tridiagonal coefficient matrix

$$\begin{bmatrix} -\frac{2}{h^2}+q(x_1) & \frac{1}{h^2}\ \frac{p(x_1)}{2h} & 0 & \cdots & & \\ & \ddots & \ddots & \ddots & & \\ \cdots\ 0 & \frac{1}{h^2}-\frac{p(x_j)}{2h} & -\frac{2}{h^2}+q(x_j) & \frac{1}{h^2}+\frac{p(x_j)}{2h} & 0 \cdots & \\ & & \ddots & \ddots & \ddots & \\ & & & & \frac{1}{h^2}-\frac{p(x_n)}{2h} & -\frac{2}{h^2}+q(x_n) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{j-1} \\ u_j \\ u_{j+1} \\ \vdots \\ u_n \end{bmatrix} = \text{r.h.s.}$$

$$\text{r.h.s} = \begin{bmatrix} f(x_1) - \left[\frac{1}{h^2} - \frac{p(x_1)}{2h}\right]\alpha \\ f(x_2) \\ \vdots \\ f(x_j) \\ \vdots \\ f(x_{n-1}) \\ f(x_n) - \left[\frac{1}{h^2} + \frac{p(x_n)}{2h}\right]\beta \end{bmatrix}$$

Questions which naturally arise:

(A) is the matrix non-singular?

YES: tridiagonal $\Rightarrow$ banded elimination

NO: infinitely many or no solution

(B) is u_j anything like $u(x_j)$? :

error $\quad e_j := u(x_j) - u_j$

In this order : A : tridiagonal elimination

$$\begin{bmatrix} b_1 & c_1 & & & \\ \boxed{a_2} & b_2 & c_2 & & \\ & \boxed{a_3} & b_3 & c_3 & \\ & & \ddots & \ddots & c_{n-1} \\ & & & \boxed{a_n} & b_n \end{bmatrix}$$

For $i = 2, \dots, n$

row $i \leftarrow$ row $i - \frac{a_i}{b_{i-1}}$ row $i-1$

end

renders an upper triangular system of linear equations easily solved in reverse order (backwards substitution) provided diagonal entries are always non-zero.

Consider simplest case : $p = 0 = q$: eqns are

$$\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & & & 1 \\ & & & 1 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ \vdots \\ f_n \end{bmatrix}$$

$f_1 = h^2 f(x_1) - \alpha$, $f_2 = h^2 f(x_2)$

$f_n = h^2 f(x_n) - \beta$

1st step

$$\begin{bmatrix} -2 & 1 & 0 & \cdots & & \\ 0 & -\frac{3}{2} & 1 & 0 & \cdots & \\ 0 & 1 & -2 & 1 & 0 & \cdots \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 + \frac{1}{2}f_1 \\ f_3 \\ \vdots \end{bmatrix}$$

2nd step

$$\begin{bmatrix} -2 & 1 & 0 & \cdots & & & \\ 0 & -\frac{3}{2} & 1 & 0 & \cdots & & \\ 0 & \boxed{0} & -\frac{4}{3} & 1 & 0 & \cdots & \\ 0 & 0 & 1 & -2 & 1 & 0 & \cdots \end{bmatrix} \qquad \begin{matrix} f_1 \\ f_2 + \frac{1}{2}f_1 \\ f_3 + \frac{2}{3}\left(f_2 + \frac{1}{2}f_1\right) \end{matrix}$$

$\vdots$

nth step

$$\begin{bmatrix} -2 & 1 & & & & \\ & -\frac{3}{2} & 1 & & & \\ & & -\frac{4}{3} & 1 & & \\ & & & -\frac{5}{4} & 1 & \\ & & & & \ddots & 1 \\ & & & & & -\frac{n+1}{n} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} f_1 \\ \hat{f}_2 \\ \vdots \\ \hat{f}_n \end{bmatrix}$$

$$\hat{f}_n = f_n + \frac{n-1}{n}\hat{f}_{n-1}.$$

and now $\quad u_n = -\frac{n}{n+1}\hat{f}_n$

$$u_{n-1} = \frac{n-1}{n}\left(\hat{f}_{n-1} - u_n\right)$$

$\vdots$

This is a special case of Gauss Elimination (G.E.)

$A = \{a_{ij}\} \in \mathbb{R}^{n\times n}$, $Ax = b$

For columns $j = 1, \dots, n-1$
  For rows $i = j+1, \dots n$
    row $i \leftarrow$ row $i - \frac{a_{ij}}{a_{jj}}$ row j (multiplier)
    $b_i \leftarrow b_i - \frac{a_{ij}}{a_{jj}} b_j$
  end
end

in which the second loop is just $i = j+1$ to $j+1$.

G.E. reduces a linear system to upper triangular form from which backwards substitution can be used to calculate the solution