Corollary (Minimum Principle)

If $\phi_{j+1} - 2\phi_j + \phi_{j-1} \le 0$, $j = 1, \dots, n$

then for each j $\quad \phi_j \geq \min\{0, \phi_0, \phi_{n+1}\}$

Proof apply lemma above to $-\phi_j$

Now consider the non-negative mesh function

$$\psi_j = \left(x_j - \frac{a+b}{2}\right)^2 \quad \text{then}$$

$$\begin{aligned} \psi_{j+1} - 2\psi_j + \psi_{j-1} &= x_{j+1}^2 - (a+b)x_{j+1} \\ &\quad - 2\left(x_j^2 - (a+b)x_j\right) \\ &\quad + x_{j-1}^2 - (a+b)x_{j-1} \\ &= (x_j + h)^2 - 2x_j^2 + (x_j - h)^2 = 2h^2 \end{aligned}$$

and let $\phi_j = e_j + \frac{1}{2}\tau\psi_j$, $j = 0, 1, \dots, n+1$

then

$$\phi_{j+1} - 2\phi_j + \phi_{j-1} = \underbrace{e_{j+1} - 2e_j + e_{j-1}}_{h^2\tau_j} + \frac{1}{\cancel{2}}\tau\cancel{2}h^2 \geq 0$$

(because recall $\tau = \max|\tau_j| \le Mh^2$)

$$\text{so } e_j + \tfrac{1}{2}\tau\psi_j \le \max\{0, \overset{0}{\cancel{e_0}} + \tfrac{1}{2}\tau\psi_0, \overset{0}{\cancel{e_{n+1}}} + \tfrac{1}{2}\tau\psi_{n+1}\} = \frac{(b-a)^2}{8}\tau$$

by max principle

but $\frac{1}{2}\tau\psi_j \geq 0$ so this certainly implies $\quad e_j \le \dfrac{(b-a)^2}{8}\tau$

Similarly taking $\phi_j = e_j - \frac{1}{2}\tau\psi_j$

$$\phi_{j+1} - 2\phi_j + \phi_{j-1} = h^2\tau_j - \frac{1}{\cancel{2}}\tau\cancel{2}h^2 \le 0$$

$$\text{so } e_j - \tfrac{1}{2}\tau\psi_j \geq -\frac{(b-a)^2}{8}\tau$$

and $-\frac{1}{2}\tau\psi_j \le 0$ so $e_j \geq -\dfrac{(b-a)^2}{8}\tau$ by min principle.

The outcome is that

$$-\frac{(b-a)^2}{8}Mh^2 \le u(x_j) - u_j \le \frac{(b-a)^2}{8}Mh^2$$

so $|u(x_j) - u_j| \to 0$ as $h \to 0$ like h^2

or $|e_j| = O(h^2)$

More general problems : (still linear) for $p > 0$

$$(pu')' + qu = f, \quad u(a) = \alpha,\ u(b) = \beta$$

$$(pu')' \simeq \frac{pu'|_{x_{j+\frac{1}{2}}} - pu'|_{x_{j-\frac{1}{2}}}}{h}$$

$(x_{j+\frac{1}{2}} = x_j + \frac{1}{2}h$, etc$)$, $\quad u'|_{x_{j+\frac{1}{2}}} \simeq \dfrac{u_{j+1} - u_j}{h}$

leads to

$$\left(p(x_{j+\frac{1}{2}})\frac{u_{j+1} - u_j}{h} - p(x_{j-\frac{1}{2}})\frac{u_j - u_{j-1}}{h}\right)\Big/h$$

$$= \underbrace{\frac{p(x_{j+\frac{1}{2}})u_{j+1} - \left[p(x_{j+\frac{1}{2}}) + p(x_{j-\frac{1}{2}})\right]u_j + p(x_{j-\frac{1}{2}})u_{j-1}}{h^2}}$$

and denote $L_h u_j = \flat + q(x_j)u_j$

$j = 1, \dots, n$
$u_0 = \alpha$, $u_{n+1} = \beta$

so finite difference equations are

$$L_h u_j = f(x_j), \quad j = 1, \dots, n$$

$u_0 = \alpha$, $u_{n+1} = \beta$

and as for the special case above <u>define</u> the local truncation error τ_j by

$$L_h u(x_j) = f(x_j) + \tau_j$$

$j = 1, \dots, n$

subtract $\quad L_h e_j = \tau_j$, $e_0 = 0 = e_{n+1}$

and if p, q are such that

$$L_h\phi_j \geq 0 \Rightarrow \phi_j \le \max\{0, \phi_0, \phi_{n+1}\}$$

$$L_h\phi_j \le 0 \Rightarrow \phi_j \geq \min\{0, \phi_0, \phi_{n+1}\}$$

then as above for the special case if there is a mesh function such that

$$L_h\psi_j = Kh^2 \Rightarrow |e_j| \le C\tau = O(h^2)$$

May or may not be so easy to establish a max/min principle or find ψ_j depending on p, q

Remark : note $L_h e_j = \tau_j$

L_h represents the rows of the coefficient matrix (called A above)

so $A e = \tau \iff e = A^{-1}\tau$

$e = (e_1, \dots, e_n)^T$, $\tau = (\tau_1, \dots, \tau_n)^T$

so $\|e\|_2 \le \|A^{-1}\|_2 \|\tau\|_2$

so an alternative way to do the error analysis would be to try to find a uniform bound for $\|A^{-1}\|_2$

In particular when A is symmetric

$$\|A\|_2 = |\lambda_{max}(A)|$$

$$\text{so } \|A^{-1}\|_2 = 1/|\lambda_{min}(A)|$$

$\lambda_{max}(A)$ = eigenvalue of A furthest from the origin

$\lambda_{min}(A)$ = " " nearest to the origin

and hence showing that $|\lambda_{min}(A)|$ is bounded below away from 0 independently of $h \Rightarrow \quad \|e\|_2 \le C\|\tau\|_2$

where $\tau_j = O(h^2)$ each j

i.e. a normwise estimate rather than a componentwise estimate as for max. principle

Example $\quad u'' = f$, $u(a) = \alpha$, $u(b) = \beta$

$$L_h \text{ `='} A = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & -2 \end{bmatrix}$$

has eigenvectors v^k with entries

$$v_j^k = \sin\frac{jk\pi}{n+1}, \quad j = 1, \dots, n$$

corresponding to eigenvalues

$$\lambda^k = \frac{1}{h^2}\left[2\cos\frac{k\pi}{n+1} - 2\right]$$

for $k = 1, \dots, n$

Proof $\quad v_0^k = v_{n+1}^k = 0$ each k and

$$\sin\frac{(j-1)k\pi}{n+1} - 2\sin\frac{jk\pi}{n+1} + \sin\frac{(j+1)k\pi}{n+1}, \quad j = 1, \dots, n$$

$$= \sin\frac{jk\pi}{n+1}\left[\cos\frac{k\pi}{n+1} - 2 + \cos\frac{k\pi}{n+1}\right]$$

$$\Rightarrow \lambda^k = \frac{1}{h^2}\left[2\cos\frac{k\pi}{n+1} - 2\right]$$