$u'' = f$, $\begin{matrix} u(a)=\alpha \\ u(b)=\beta \end{matrix}$ $\Rightarrow$ matrix A

eigenvalues :

$$\lambda^k = \frac{1}{h^2}\left[2\cos\frac{k\pi}{n+1} - 2\right], \quad k=1,\dots,n$$

$[(n+1)h = b-a]$

so $$\lambda_{max} = \frac{1}{h^2}\left[2\cos\frac{n\pi}{n+1} - 2\right]$$

$$= -4h^{-2} + O(1)$$

$$\lambda_{min} = \frac{1}{h^2}\left[2\underbrace{\cos\frac{\pi}{n+1}}_{1-\frac{1}{2}\frac{\pi^2}{(n+1)^2}+\dots} - 2\right]$$

$$\simeq -\frac{\pi^2}{h^2(n+1)^2} + O(n^{-4})$$

$$= -\frac{\pi^2}{(b-a)^2} + O(n^{-4})$$

so $|\lambda_{min}(A)| = \frac{\pi^2}{(b-a)^2} + \dots$ is a constant independent of h

$$\Rightarrow \|e\|_2 = \left[\sum_{j=1}^{n}(u(x_j)-u_j)^2\right]^{1/2}$$

$$\leq \frac{(b-a)^2}{\pi^2}\left(\sum_{j=1}^{n}\tau_j^2\right)^{1/2}, \quad \tau_j = O(h^2)$$

Also useful in this context (and others as we shall see) is

Gershgorin's Theorem

If λ is an eigenvalue of
$$A = \{a_{ij} : i,j = 1,\dots,n\} \in \mathbb{R}^{n\times n}$$
then λ lies in at least one of the (Gershgorin) discs
$$\left\{ z \in \mathbb{C} : |a_{ii} - z| \leq \sum_{\substack{j=1\\j\neq i}}^{n} |a_{ij}| \right\} \quad i=1,\dots,n$$

Proof λ an eigenvalue $\Rightarrow \exists\, x \neq 0$ with $Ax = \lambda x$ or $\sum_{j=1}^{n} a_{ij}x_j = \lambda x_i$, $i=1,\dots,n$.

Suppose $|x_k| \geq |x_i|$, $i=1,\dots,n$ then
$$\sum_{j=1}^{n} a_{kj}x_j = \lambda x_k$$
$$\Leftrightarrow (a_{kk}-\lambda)x_k = -\sum_{\substack{j=1\\j\neq k}}^{n} a_{kj}x_j$$
$$\Rightarrow |a_{kk}-\lambda|\,|x_k| \leq \sum_{\substack{j=1\\j\neq k}}^{n} |a_{kj}|\,|x_j|$$
$$\Leftrightarrow |a_{kk}-\lambda| \leq \sum_{\substack{j=1\\j\neq k}}^{n} |a_{kj}|\left|\frac{x_j}{x_k}\right|$$
$$\leq \sum_{\substack{j=1\\j\neq k}}^{n} |a_{kj}|$$

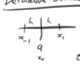
Example $u'' - u = f$

$$A = \begin{bmatrix} \ddots & & \\ \frac{1}{h^2} & -\frac{2}{h^2}-1 & \frac{1}{h^2} \\ & & \ddots \end{bmatrix}$$

Gershgorin's Th $\Rightarrow$ for any eigenvalue λ
$$\left|+\frac{2}{h^2}+1+\lambda\right| \leq \frac{2}{h^2}$$
$$\Rightarrow 1+\lambda \leq 0 \quad \text{i.e. } \lambda \leq -1$$
so $|\lambda_{min}| \geq 1$
$$\Rightarrow \|e\|_2 \leq \|\tau\|, \quad \tau_j = O(h^2) \text{ in above.}$$

Remark: Gershgorin's Theorem is one of the few techniques for establishing invertibility : see problem sheet for another

Derivative boundary conditions e.g. $u'(a) = \alpha$



introduce a ficticious point $x_{-1} = a - h$ and approximate with the central difference
$$\frac{u_1 - u_{-1}}{2h} = \alpha$$
or $u_1 - 2h\alpha = u_{-1}$ — ①

then e.g. for $u'' = f$ we use the finite difference centred on x_0 :
$$u_1 - 2u_0 + u_{-1} = h^2 f_0.$$
Use ① to eliminate u_{-1} $\Rightarrow$
$$\Rightarrow 2u_1 - 2u_0 = h^2 f_0 + 2h\alpha$$
or to preserve symmetry
$$u_1 - u_0 = \tfrac{1}{2}h^2 f_0 + h\alpha$$

$$\Rightarrow \begin{bmatrix} -1 & 1 & & \\ 1 & -2 & 1 & \\ & \ddots & \ddots & \ddots \\ & & & \end{bmatrix}\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} \frac{1}{2}h^2 f_0 + h\alpha \\ h^2 f_1 \\ \vdots \\ \vdots \end{bmatrix}$$

Note if $u'' = f$ with $u'(a) = \alpha$, $u'(b) = \beta$ then

$$\begin{bmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -1 \end{bmatrix}\begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ \vdots \\ u_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}h^2 f_0 + h\alpha \\ h^2 f_1 \\ \vdots \\ \vdots \\ \frac{1}{2}h^2 f_{n+1} - h\beta \end{bmatrix}$$

note now this is singular since

$$\begin{bmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & & \ddots & & \\ & & 1 & -2 & 1 \\ & & & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ \vdots \\ \vdots \\ 1 \end{bmatrix} = \underline{0}$$

i.e. given $f_i = 0$ all i and $\alpha = 0 = \beta$ so r.h.s. $= \underline{0}$, there is a non-zero solution
$$u_0 = u_1 = \dots u_{n+1} = 1$$
but this corresponds exactly to the ODE BVP : $u'' = 0$
$$u'(a) = 0 = u'(b)$$
since $u =$ constant satisfies this homogeneous BVP and hence any constant can be added to any particular solution u of $u'' = f$, $u'(a) = \alpha$, $u'(b) = \beta$.

'Mixed' boundary conditions
$$cu(a) + du'(a) = \alpha$$
can also be approximated by
$$cu_0 + d\,\frac{u_1 - u_{-1}}{2h} = \alpha$$
and u_{-1} eliminated from
$$u_1 - 2u_0 + u_{-1} = h^2 f_0$$
(or $p_{\frac{1}{2}}u_1 - [p_{\frac{1}{2}} + p_{\frac{3}{2}}]u_0 + p_{-\frac{1}{2}}u_{-1} = h^2 f_0$ for $(pu')' = f$ etc.

Finite differences for higher order problems

e.g. beam problem $u^{iv} = f$

as an alternative to introducing $z = u''$ can approximate u^{iv} directly:
$$u(x_j \pm 2h) = u \pm 2hu' + \frac{4h^2}{2}u'' \pm \frac{8h^3}{6}u''' + \frac{16h^4}{24} + \dots$$
$$+O(h^5)$$
$(u = u(x_j),\ u' = u'(x_j)$ etc $)$
$$\Rightarrow u(x_j+2h) + u(x_j-2h) = 2u + 4h^2u'' + \tfrac{4}{3}h^4u^{iv} + O(h^6)$$
and $$u(x_j+h) + u(x_j-h) = 2u + h^2u'' + \frac{h^4}{12}u^{iv} + O(h^6)$$
so
$$\frac{u_{j+2} - 4u_{j+1} + 6u_j - 4u_{j-1} + u_{j-2}}{h^4} \sim u^{iv}$$
with $\tau_j = O(h^2)$

Boundary conditions:
$$u(x_0) = \alpha,\ u'(x_0) = \alpha',\ u(x_{n+1}) = \beta,\ u'(x_{n+1}) = \beta'$$
$u(x_0)=\alpha \Downarrow u_0 = \alpha$; $u'(x_0)=\alpha' \Downarrow$
$$\frac{u_1 - u_{-1}}{2h} = \alpha'$$
$\Rightarrow u_{-1} = u_1 - 2h\alpha'$ — ①

$u(x_{n+1})=\beta \Downarrow u_{n+1} = \beta$; $u'(x_{n+1})=\beta' \Downarrow$
$$\frac{u_{n+2} - u_n}{2h} = \beta'$$
$\Rightarrow u_{n+2}$ similarly — ②

so
$$\frac{u_{j+2} - 4u_{j+1} + 6u_j - 4u_{j-1} + u_{j-2}}{h^4} = f(x_j) \quad j = 1,2,\dots,n$$
with u_{-1} given by ① for $j=1$