Higher order method: we have seen so far finite difference method that give $\tau_j = O(h^2)$ i.e. 2nd order accurate which has implied $e_j = O(h^2)$ i.e. 2nd order convergent for certain problems. Can one get more accurate results with high order finite differences?

$$-\tfrac{1}{12}u(x_{j+2}) + \tfrac{4}{3}u(x_{j+1}) - \tfrac{5}{2}u(x_j) + \tfrac{4}{3}u(x_{j-1}) - \tfrac{1}{12}u(x_{j-2})$$
$$= h^2 u''(x_j) + O(h^6)$$
(see problem sheet)

so
$$\frac{-\tfrac{1}{12}u_{j+2} + \tfrac{4}{3}u_{j+1} - \tfrac{5}{2}u_j + \tfrac{4}{3}u_{j-1} - \tfrac{1}{12}u_{j-2}}{h^2} = f(x_j) \qquad (*)$$

has $\tau_j = O(h^4)$ for $u'' = f$

Note a difficulty: $u(a) = \alpha$, $u(b) = \beta$ only are given, so $(*)$ for $j=1$ and $j=n$ has u_{-1} and u_{n+2}: outside the domain: need also special near boundary treatments something like $\dfrac{u_{-1}+u_1}{2} = \alpha$ etc which might reduce accuracy. These methods are not so popular, but non-centred (one-sided) $O(h^4)$ difference schemes for $j=1$ and $j=n$ could be derived.

All the finite differences we have considered so far are <u>centred</u> or <u>central differences</u> since they are symmetric about the central point, x_j, for even order derivatives and anti-symmetric for odd order derivatives.

Any scheme for a given problem which has $\tau_j \to 0$ as $h \to 0$ is said to be <u>consistent</u>

Nonlinear BVPs

$$u'' = g(x, u, u')$$
$$u(a) = \alpha, \; u(b) = \beta$$

Finite difference replacement:

$$\frac{u_{j+1} - 2u_j - u_{j-1}}{h^2} = g\left(x_j, u_j, \frac{u_{j+1} - u_{j-1}}{2h}\right)$$
$$j = 1, \dots, n$$
$$u_0 = \alpha, \; u_{n+1} = \beta$$

is now a system of nonlinear equations to which Newton iteration or any other nonlinear solution method could be applied

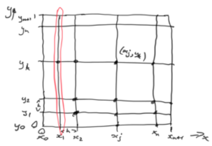
2-D Elliptic BVPs.

Model problem:

$$-\nabla^2 u = -u_{xx} - u_{yy}$$
$$= -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f$$

$$(x,y) \in (0,1) \times (0,1) = \Omega,$$
the domain

u given on $\partial\Omega$: namely the boundary $x = 0, 1$, $y = 0, 1$ (Dirichlet Boundary conditions)



$$U_{jk} \simeq u(x_j, y_k), \quad j,k = 1, \dots, n$$

u_{0k}, $u_{n+1\,k}$, u_{j0}, $u_{j\,n+1}$ all given by the boundary conditions

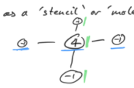
2nd order accurate finite difference replacement in each direction

$$\frac{-u_{j+1\,k} + 2u_{jk} - u_{j-1\,k}}{h^2} + \frac{-u_{j\,k+1} + 2u_{jk} - u_{j\,k-1}}{h^2} = f(x_j, y_k)$$

or
$$4u_{jk} - u_{j+1\,k} - u_{j-1\,k} - u_{j\,k+1} - u_{j\,k-1} = h^2 f(x_j, y_k)$$
$$j,k = 1, \dots, n$$

Note: $n^2 \times n^2$ linear system of equations

This is the <u>5-point</u> formula, sometimes written as a 'stencil' or 'molecule'



With bottom ↗ top then left → right (lexicographic) ordering so that

$$\underline{u} = (u_{11}, u_{12}, \dots, u_{1n}; u_{21}, u_{22}, \dots, u_{2n}; \dots; u_{n1}, u_{n2}, \dots, u_{nn})^T$$

is the vector of unknowns, the resulting matrix system is of the form

$$A\underline{u} = \underline{f}$$

with $\underline{f}$ being values of $f(x_j, y_k)$ and taking into account the boundary conditions for $j,k = 0, n$

and
$$A = \frac{1}{h^2}\begin{bmatrix} B & C & 0 & \dots & 0 \\ C & B & C & 0 & \dots 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & C \\ 0 & \dots & 0 & C & B \end{bmatrix} \in \mathbb{R}^{n^2 \times n^2}$$

$$B = \begin{bmatrix} 4 & -1 & & 0 \\ -1 & 4 & -1 & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 4 \end{bmatrix} \in \mathbb{R}^{n \times n}, \quad C = -I \in \mathbb{R}^{n \times n}$$

A is the <u>5-point matrix</u>.

$0 \dots 0 \; -1 \; 0 \dots 0 \; -1 \; 4 \; -1 \; 0 \dots 0 \; -1 \; 0 \dots 0$