$$A = \frac{1}{h^2}\begin{bmatrix} B & -I & & \\ -I & B & \ddots & \\ & \ddots & \ddots & -I \\ & & -I & B \end{bmatrix} \in \mathbb{R}^{n^2\times n^2}, \quad B = \begin{bmatrix} 4 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 4 \end{bmatrix} \in \mathbb{R}^{n\times n}$$

Note: A is symmetric $\Rightarrow$ all eigenvalues are real, so by Gershgorin's Theorem

$$0 \le \lambda \le 8h^{-2}$$

for all eigenvalues λ.

But Gershgorin's Theorem can be generalised to prove $0 < \lambda$.

Definition A square matrix A is **reducible** if $\exists$ a permutation matrix with $P^TAP = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$ with A_{11}, A_{22} square

[$\Rightarrow$ linear system

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$ can be

solved in 2 parts: $A_{22}x_2 = f_2$ for x_2

then $A_{11}x_1 = f_1 - A_{12}x_2$ for x_1]

If A is not reducible then it is **irreducible**

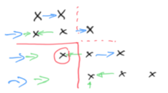
Lemma If a matrix $A \in \mathbb{R}^{m\times m}$ is irreducible then for any 2 indices k, ℓ there exist indices $j_1, j_2, \dots, j_r$ for some finite r with $a_{kj_1} \neq 0, a_{j_1j_2} \neq 0, a_{j_2j_3} \neq 0, \dots, a_{j_r\ell} \neq 0$.
In particular any one index is connected to all others by some such path

Proof suppose there exist k, ℓ such that no j_i's satisfying the above exist. Suppose $a_{ki_2}, a_{ki_3}, \dots, a_{ki_s}$ are the non-zero entries of row k, and similarly consider all indices of non-zero entries of rows $i_2, \dots, i_s$ and continue to find the set of all indices $\{i\}$ for which there are j_i's as above connecting from k. Then permute so that these are the last indices $m, m-1, \dots, m-r$, hence

$$P^TAP = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$ i.e. reducible

because $m-r>1$ since ℓ is not in this set

Example (i) A tridiagonal matrix is irreducible if and only if all of its super-diagonal and subdiagonal entries are non-zero



(ii) the 5-point matrix is irreducible

Theorem (extension of Gershgorin's Theorem for irreducible matrices)
If $A \in \mathbb{R}^{m\times m}$ is irreducible and an eigenvalue λ lies on the boundary (the edge) of the union of all of the Gershgorin discs

$$D_i = \{z \in \mathbb{C} \mid |a_{ii} - \lambda| \le \sum_{\substack{j=1\\ j\neq i}}^{m} |a_{ij}|\}$$

then it must lie on the boundary of each of the Gershgorin discs $D_1, D_2, \dots, D_m$.

Proof λ an eigenvalue $\Rightarrow \exists x \neq 0 : Ax = \lambda x$.
Suppose $|x_k| \ge |x_i|$ each i and normalise x so that $x_k = 1$ ($\Rightarrow |x_j| \le 1$ all j).

$$Ax = \lambda x \Leftrightarrow (a_{kk} - \lambda)x_k = \sum_{\substack{j=1\\ j\neq k}}^{m} a_{kj}x_j$$

$$\Rightarrow |a_{kk} - \lambda| \le \sum_{\substack{j=1\\ j\neq k}}^{m} |a_{kj}||x_j| \le \sum_{\substack{j=1\\ j\neq k}}^{m} |a_{kj}| \quad (*)$$

but λ is on the boundary of $\bigcup_i D_i$ so it can not be in the interior of D_k

$$\Rightarrow |a_{kk} - \lambda| = \sum_{\substack{j=1\\ j\neq k}}^{m} |a_{kj}|$$

in $(*)$ $\Rightarrow |x_j| = 1$ for each j for which $a_{kj} \neq 0$. By irreducibility $\exists$ at least one index $p \neq k$ for which $a_{kp} \neq 0$ and so $|x_p| = 1$. Repeat with p instead of k

$$|a_{pp} - \lambda| \le \sum_{\substack{j=1\\ j\neq p}}^{m} |a_{pj}||x_j| \le \sum_{\substack{j=1\\ j\neq p}}^{m} |a_{pj}| = |a_{pp} - \lambda|$$

since λ is on the boundary of $\bigcup_i D_i$

i.e. λ lies on the boundary of D_p

$\Rightarrow |x_j| = 1$ for each j for which $a_{pj} \neq 0$.

Because of irreducibility this process can be continued through all rows

Corollary The 5-point matrix is nonsingular

Proof it is irreducible and not all of the Gershgorin discs touch the origin

4 -1 0 … 0 -1 0 … (1^{st} row)

More generally:

Definition $A \in \mathbb{R}^{m\times m}$ is <u>irreducibly diagonally dominant</u> (IDD) if it is irreducible and satisfies

$$|a_{ii}| \ge \sum_{\substack{j=1\\ j\neq i}}^{m} |a_{ij}|, \quad i = 1, \dots, m$$

with strict inequality for at least one i

Corollary If A is IDD then it is nonsingular. Moreover if $a_{ii} > 0$ each i then all of the eigenvalues lie in the right half plane. If A is symmetric and IDD then it is positive definite (i.e. all eigenvalues are real and positive).